Tesis Doctoral / Ph.D. Thesis

Theoretical and numerical aspects for nonlocal equations of porous medium type

Author:
Félix del Teso Méndez

Advisor:
Dr. Juan Luis Vázquez Suárez

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La vida no es fácil.

Abstract

Theoretical and numerical aspects
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by Félix DEL TESO MÉNDEZ

In this thesis we consider three different models of nonlinear and nonlocal diffusion equations of porous medium type. The prototype is the classical Porous Medium Equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad \text{(PME)}$$

which models the flow of gasses through a porous media.

My contribution to this field is contained in the following papers: [40, 41, 45, 69–71] in collaboration with my advisor, Professor Juan Luis Vázquez, and my colleagues Diana Stan, Espen R. Jakobsen and Jørgen Endal.

The main topics of the thesis are the following:

(1) *Finite difference method for the Fractional Porous Medium Equation* (FPME) $u_t + (-\Delta)^s u^m = 0$ with $m \geq 1$ and $s \in (0, 1)$. In [40] we propose a convergent numerical scheme based on the Caffarelli-Silvestre extension method for the case $s = 1/2$. In [41] we discuss the general case $s \in (0, 1)$.

(2) *Finite and infinite speed of propagation for the Porous Medium Equation with Fractional Pressure* (PMEFP) $u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$ with $m \geq 1$ and $s \in (0, 1)$. This problem has been recently studied by Caffarelli and Vázquez in [25] when $m = 2$. In [69, 70] we show the effect of the nonlinearity on the speed of propagation of the solution. To be more precise, we prove that for $m \in (1, 2)$ the solution has infinite speed of propagation, while for $m \geq 2$ the solution has finite speed of propagation.

(3) *Transformations of self-similar solutions for porous medium equations of fractional type.* In [71] we show explicit transformation for a certain kind of self-similar solutions of models (FPME) and (PMEFP). Moreover, transformations for solutions of other fractional diffusion models are studied.

(4) *Uniqueness and properties of distributional solutions of the more general nonlocal porous medium equation* (VGL) $\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0$ where $\mathcal{L}^\mu$ is a very general nonlocal operator and the nonlinearity φ is continuous and nondecreasing scalar function. In [45] we prove uniqueness, existence, stability and compactness for this model. Moreover we propose convergent numerical schemes.

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Nunca he sido muy de expresar sentimientos a la gente, pero dado que una Tesis doctoral solo se hace (con suerte) una vez en la vida aprovecho para ponerme un poco sentimental. Aunque la mayoría de la gente de la que hablaré aquí no han sido parte activa de los resultados de mi investigación, sí que han sido parte de algo todavía más importante, MI VIDA.

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[1]

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Félix

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Dedicado a MI MADRE,
que pese a todas la dificultades,
nos consiguió sacar adelante…

Introducción

La ecuación del calor es el modelo más sencillo y conocido de proceso de difusión,

$$\frac{\partial u}{\partial t} = \Delta u.$$

La función $u = u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ denota la concentración o densidad de una cierta sustancia que evoluciona en un dominio a lo largo del tiempo. La descripción original de esta ecuación fue dada por Joseph Fourier en su libro de 1822 titulado "Théorie Analytique de la Chaleur". Existe también una estrecha relación de la ecuación del calor y el movimiento Browniano de partículas, aparecido por primera vez en el famoso artículo de Albert Einstein [44]. Cabe mencionar la versión discreta de la ecuación del calor, también conocida como camino aleatorio o paseo del borracho, que describe un proceso de difusión lineal en un conjunto discreto. En particular, bajo unas ciertas condiciones en la discretización, se puede obtener la ecuación del calor como límite del camino aleatorio cuando la distancia entre los puntos de la discretización tiende a cero.

Por suerte o por desgracia, no todos los fenómenos físicos se pueden describir por ecuaciones en derivadas parciales lineales. Por esta razón, las ecuaciones en derivadas parciales no lineales se han convertido en una herramienta esencial de la ciencia moderna que se utiliza para describir numerosos fenómenos físicos como la mecánica de fluidos, dinámica de poblaciones, relatividad, termodinámica, etc. Una clase importante de ecuaciones en derivadas parciales no lineales son las ecuaciones parabólicas de segundo orden, usadas frecuentemente para describir procesos de convección y difusión para problemas enmarcados en los campos anteriormente mencionados.

El estudio de las ecuaciones en derivadas parciales no lineales es muy complejo. Hasta la fecha no existe una estrategia unificada con la que abordar los problemas que aparecen naturalmente como pueden ser la búsqueda de soluciones explícitas, existencia, unicidad y regularidad de soluciones, comportamiento asintótico, velocidad de propagación o aproximaciones numéricas. En esta tesis doctoral se tratan muchas de estas cuestiones para un tipo de ecuaciones de segundo orden con difusión no lineal llamadas ecuaciones de medios porosos.

0.1 Motivación y aplicaciones físicas

0.1.1 El modelo físico

La ecuación en la que se basan los modelos que abarcan esta tesis fue derivada de manera independiente por Leibenzon y Muskat allá en el año 1930 mientras se encontraban estudiando el flujo de un gas en un medio poroso. Unos años antes, en 1903, utilizando una ley física que acabaría llevando su nombre, Boussinesq también había considerado este modelo para el estudio de filtraciones de aguas subterráneas.

Se considera un medio continuo que bien puede ser un líquido o una población suficientemente grande y se representa a través de una densidad de distribución $u(x,t) \geq 0$ que evoluciona en el tiempo regido por un campo de velocidades $v(x,t)$ que a su vez también puede variar con el tiempo. La correspondiente ecuación de continuidad viene dada por

$$\frac{\partial u}{\partial t} + \nabla \cdot (u \cdot v) = 0.$$

Se hacen ahora dos suposiciones físicas:

(a) La velocidad de los fluidos en un medio poroso se comporta conforme a la Ley de Darcy, esto es, v es un potencial de la forma

$$v = -\nabla p,$$

donde p denota la presión a la que está sometido el fluido.

(b) Existe una relación entre la presión p y la densidad del fluido u: Leibenzon y Muskat obtuvieron una relación de la forma

$$p = f(u),$$

donde f es una función escalar y no decreciente. Cuando se trata de un flujo isotérmico la función f es lineal ,mientras que para flujos del tipo adiabáticos se obtienen potencias de orden superior del tipo $f(u) = cu^{m-1}$ con $c > 0$ y $m > 1$.

La dependencia lineal de f con respecto a u fue obtenida también por Boussinesq y utilizada en la ecuación $u_t = c\nabla \cdot (u\nabla u) = \frac{c}{2}\Delta u^2$, que servía como modelo para la filtración de agua en una capa de tierra horizontal.

Salvo constantes, la clásica Ecuación de Medios Porosos se puede escribir como $u_t = \nabla \cdot (u^{m-1}\nabla u)$, o en su forma más conocida,

$$\frac{\partial u}{\partial t} = \Delta u^m.$$

Este típico modelo de ecuación no lineal ha sido investigado por muchos autores. Para una referencia muy completa sobre las propiedades de la Ecuación de Medios Porosos, nos remitimos al libro [77] de Juan Luis Vázquez.

0.1.2 Difusión no lineal

Como se ha comentado, uno de los modelos más populares de difusión no lineal es la Ecuación de Medios Porosos,

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{para} \qquad m > 1. \tag{PME}$$

La (PME) pertenece a la clase más amplia de ecuaciones parabólicas del tipo

$$\frac{\partial u}{\partial t} = \nabla \cdot (A(u)\nabla u),$$

donde $A(u)$ suele ser llamado coeficiente de difusión. En el caso de la (PME) se tiene que $A(u) = u^{m-1}$, lo que hace que la ecuación sea degenerada para $m > 1$. También se estudia la ecuación (PME) cuando $m < 1$ y por lo tanto se convierte en singular. Este último caso recibe el nombre de Ecuación de Difusión Rápida. John R. King consideró la Ecuación de Difusión Rápida en [61] para modelar la difusión de impurezas en el silicio.

La terminología *difusión rápida/lenta* se refiere a la propiedad de velocidad de propagación finita o infinita de las soluciones. Eso quiere decir, que cuando $m > 1$, si el dato inicial tiene soporte compacto, las soluciones de (PME) tienen soporte compacto para todo tiempo $t > 0$. En particular esto abre el campo de las llamadas *fronteras libres*, en el que se estudian las propiedades de las fronteras del dominio donde las soluciones son distintas de cero. Por otro lado, cuando $m \leq 1$, las soluciones de (PME) tienen velocidad de propagación infinita, es decir, dado un dato inicial no negativo cualquiera (posiblemente de soporte compacto), la solución automáticamente es estrictamente positiva en todo el espacio para cualquier tiempo $t > 0$.

0.1.3 Difusión no local

En los procesos de difusión descritos hasta ahora, solo se tiene en cuenta la interacción que tienen las partículas de manera local. En general, muchas de estas interacciones no son meramente locales y se requiere del uso de operadores más generales que el Laplaciano. El ejemplo por excelencia de operador no local es el llamado Laplaciano fraccionario (Stein [73], Landkof [63]), que tiene muchas definiciones equivalentes. Si denotamos por $\mathcal{F}$ a la Transformada de Fourier, definimos el Laplaciano fraccionario $(-\Delta)^s$ con $s \in (0,1)$ para funciones f de la clase de Schwartz como

$$\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi).$$

Esta definición generaliza de alguna manera la propiedad $\mathcal{F}(\Delta f)(\xi) = -|\xi|^2\mathcal{F}(f)(\xi)$ del Laplaciano. De manera equivalente, el operador Laplaciano fraccionario se puede definir a través de la siguiente integral singular:

$$(-\Delta)^s f(x) = C_{N,s}\text{V.P.}\int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+2s}}dy,$$

donde V.P denota el valor principal de la integral y $C_{N,s} = \pi^{-(2s+N/2)}\frac{\Gamma\left(\frac{N}{2}+s\right)}{\Gamma(-s)}$ es una constante de normalización. Formalmente, si tomamos $s \to 1^-$ recuperamos el Laplaciano local mientras que para $s \to 0^+$ se obtiene la identidad. Nos remitimos a [75] para una demostración muy ilustrativa de la equivalencia de ambas definiciones.

Existe una tercera definición equivalente que viene dada a través de el semigrupo del calor asociado el operador $-\Delta$:

$$(-\Delta)^s f(x) = \frac{1}{\Gamma(-\sigma)}\int_0^\infty \left(e^{-t(-\Delta)}f(x) - f(x)\right)\frac{dt}{t^{1+s}}.$$

Nos remitimos al artículo [74] donde se discute el uso de esta última definición.

El operador inverso del Laplaciano fraccionario se obtiene como convolución con el potencial de Riesz y para $N > 2s$ viene dado por la fórmula

$$(-\Delta)^{-s} f(x) = C_{N,-s}\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}dy.$$

En probabilidad, el Laplaciano fraccionario es el generador infinitesimal de un proceso $(s/2)$-estable, también llamados procesos estables de Lévy. En realidad el Laplaciano fraccionario es un caso particular de una clase mucho más general de operadores de difusión no local, que son generadores de procesos simétricos de Lévy. Sea μ una medida de Radon positiva y simétrica (posiblemente singular) que satisface la condición $\int_{\mathbb{R}^N} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < +\infty$. Definimos para $\psi \in C_c^\infty(\mathbb{R}^N)$ el operador generador del proceso de Lévy de difusión no local simétrico como

$$\mathcal{L}^\mu[\psi](x) = \int_{\mathbb{R}^N\setminus\{0\}} \psi(x+z) - \psi(x) - z\cdot D\psi(x)\mathbf{1}_{|z|\le 1}\,\mathrm{d}\mu(z),$$

donde $\mathbf{1}_{|z|\leq 1}$ denota la función indicatriz en el conjunto $\{|z| \leq 1\}$. Nótese que para $\mathrm{d}\mu(z) = \frac{C_{N,s}}{|z|^{N+2s}} dz$, se tiene que $\mathcal{L}^\mu = (-\Delta)^s$.

Los procesos de difusión anómala están a menudo descritos por modelos no lineales y no locales. A continuación se presentan tres modelos de difusión no local del tipo medio poroso que serán el tema principal de estudio en este trabajo.

(I) Ecuación de Medios Porosos Fraccionaria. Para $s \in (0,1)$ y $m \geq 1$ consideramos la ecuación

$$\frac{\partial u}{\partial t} + (-\Delta)^s u^m = 0. \tag{FPME}$$

Este modelo es una posible generalización en el caso no local del clásico (PME) (de hecho coinciden cuando $s = 1$). Existen grandes diferencias entre los comportamientos de las soluciones del (PME) y del (FPME). El primero y más sorprendente es que las soluciones del (FPME) tienen velocidad de propagación infinita, mientras que las del (PME) tienen velocidad de propagación finita. Esta característica del (FPME) es una consecuencia del carácter no local del operador $(-\Delta)^s$ cuando $s \in (0,1)$. Los primeros trabajos conocidos sobre el (FPME) aparecen en la serie de artículos [37–39, 82] de Arturo de Pablo, Fernando Quirós, Ana Rodríguez y Juan Luis Vázquez en el que tratan la ecuación en $\mathbb{R}^N$. Posteriormente los trabajos [16, 18] de Matteo Bonforte, Yannick Sire y Juan Luis Vázquez tratan el caso de dominios acotados. El Capítulo 1 y parte del Capítulo 5 de este manuscrito están dedicados al estudio numérico de la ecuación (FPME).

(II) Ecuación de Medios Porosos con Presión Fraccionaria. Para $s \in (0,1)$, consideramos la Ecuación de Medios Porosos escrita en forma de divergencia y asumimos que la presión p está relacionada con la densidad u por medio de un operador no local del tipo Laplaciano fraccionario inverso, esto es,

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(u \nabla (-\Delta)^{-s} u\right). \tag{CV}$$

Este modelo fue considerado for Luis Caffarelli y Juan Luis Vázquez en [25] donde demuestran existencia de soluciones y la propiedad de velocidad de propagación finita de éstas. Posteriormente, los mismos autores en colaboración con Fernando Soria estudian en [22, 23] la regularidad de las soluciones.

Uno de los principales temas de investigación de esta Tesis doctoral se basa en el estudio de la generalización del modelo (CV) para flujos más generales. En el Capítulo 2 se estudia de manera exhaustiva la velocidad de propagación de las soluciones del modelo

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(u^{m-1} \nabla (-\Delta)^{-s} u\right). \tag{PMEFP}$$

Además en el Capítulo 3 se exponen una serie de relaciones entre las soluciones de los modelos (FPME) y (PMEFP). Por último, en la parte final del Capítulo 5 se propone un posible punto de partida para el estudio numérico del (PMEFP) por medio de una generalización del problema de extensión introducido por Luis Caffarelli y Luis Silvestre en [21] adaptado a Laplacianos fraccionarios inversos.

(III) Ecuación de Medios Porosos no local generalizada. Se pueden considerar unos modelos de difusión no lineal y no local más generales que el (FPME). En el Capítulo 4 se estudian las soluciones distribucionales del problema

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0, \tag{VGL}$$

donde $\mathcal{L}^\mu$ es el operador no local general definido en la Sección 0.1.3 y la no linealidad φ es una función escalar continua y no decreciente cualquiera. Soluciones de viscosidad y de entropía para este tipo de ecuaciones han sido consideradas anteriormente por autores como Espen R. Jakobsen, Jørgen Endal, Simone Cifani o Kenneth H. Karlsen en [32, 45, 58].

0.2 Resultados principales

Se presentan a continuación los resultados principales obtenidos a lo largo de la elaboración de la presente Tesis doctoral.

0.2.1 Capítulo 1: Análisis numérico para la Ecuación de Medios Porosos Fraccionaria

Durante la preparación de los artículos citados anteriormente sobre la Ecuación de Porosos Fraccionaria,

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{1}{2}} u^m = 0 \qquad x \in \mathbb{R}^N, \ t > 0 \tag{FPME}$$

para $m \geq 1$, los autores se dieron cuenta de que en aquel momento todos los trabajos numéricos sobre esta ecuación, pasaban por el estudio de métodos de cuadratura para integrales singulares. Se propuso entonces una dirección completamente distinta. En este capítulo se propone un método numérico explícito de diferencias finitas para (FPME). La gran novedad de este trabajo reside en el tratamiento local de la ecuación no local (FPME): se presenta un método de diferencias finitas para el siguiente problema de evolución en $N+1$ dimensiones:

$$\begin{cases} \Delta w = 0, & x \in \mathbb{R}^N,\ y > 0,\ t > 0, \\ \dfrac{\partial w^{1/m}}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y}(x,0,t), & x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ w(x,0,0) = f^m(x), & x \in \mathbb{R}^N. \end{cases} \tag{0.2.1}$$

Los problemas (FPME) y (0.2.1) son equivalentes en el sentido de que $u(x,t)$ es la traza (cuando $y=0$) de $w^{1/m}(x,y,t)$. Se prueba existencia, unicidad y principio del máximo para las soluciones del método numérico. Además se prueba que dichas soluciones convergen a las soluciones del (FPME) con una velocidad del orden de $O(\Delta t + \Delta x)$ donde Δt y Δx son la longitud entre puntos de la discretización en tiempo y en espacio respectivamente. Por último se presentan experimentos numéricos de varios tipos en los que se observa que el orden de convergencia probado es el correcto.

Contemporánea y posteriormente a estos resultados sobre métodos numéricos para ecuaciones fraccionarias, otros autores se han interesado y han desarrollado aproximaciones numéricas, algunas similares y otras completamente distintas. Por citar algunas, Chen, Nochetto, Otárola y Salgado presentan en [29] un método basado en la extensión armónica (0.2.1), pero utilizando elementos finitos. Por otro lado, Cifani y Jakobsen en [33] se enfrentan directamente con la formulación no local en forma de integral singular. Huang y Oberman combinan diferencias finitas con cuadratura numérica para integrales en [54]. Por último, recientemente Ciaurri, Roncal, Stinga, Torrea y Varona presentan en [30] una idea completamente diferente a todas las mencionadas anteriormente; su método numérico se basa en aproximar el Laplaciano fraccionario por las potencias fraccionarias del Laplaciano discreto. Para ello parten de la definición del Laplaciano fraccionario forma de semigrupo del calor dada en la sección 0.1.3.

0.2.2 Capítulo 2: Ecuación de Medios Porosos con Presión Fraccionaria

Este capítulo está dedicado al estudio del problema

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot \left(u^{m-1} \nabla (-\Delta)^{-s} u\right) \qquad x \in \mathbb{R}^N,\ t > 0, \tag{PMEFP}$$

para $m > 1$, $s \in (0,1)$ y $u(x,t) \geq 0$ en dimensión $N \geq 1$. Consideramos una condición inicial

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N$$

donde $u_0 : \mathbb{R}^N \to [0,\infty)$ es una función acotada (a veces supondremos que tiene también soporte compacto o algún tipo de decaimiento rápido en el infinito). Como ya hemos comentado, en este modelo, escrito en forma de divergencia, la presión p está relacionada con la densidad u a través de un operador fraccionario. Más concretamente, $p = (-\Delta)^{-s}$ para $s \in (0,1)$. El caso particular de (PMEFP) cuando $m = 2$ fue introducido por Luis Caffarelli y Juan Luis Vázquez en [25] como un modelo de difusión no lineal del tipo medios porosos con efectos de difusión no locales.

Los resultados de este capítulo han aparecido en los artículos [69, 70] en colaboración con Juan Luis Vázquez y Diana Stan. En ellos se establecen unos resultados de existencia para cierta clase de soluciones débiles en el rango $m \in (1,3)$. Estas soluciones son el resultado de un método de aproximación por soluciones regulares. Nos referiremos a ellas como *soluciones débiles construidas*.

El objetivo principal de este trabajo era estudiar la velocidad de propagación para las soluciones de (PMEFP) motivados por el resultado probado en [25] en el que se establecía que las soluciones débiles construidas tenían velocidad de propagación finita. De esta manera descubrimos que el caso $m = 2$ marcaba un punto de inflexión en la velocidad de propagación: bajo ciertas hipótesis probamos que cuando $m \in [2,\infty)$ las soluciones tienen **velocidad de propagación finita**, mientras que para $m \in [1,2)$ tienen **velocidad de propagación infinita**.

El principal problema al que nos enfrentamos abordando este trabajo era la falta de principio de comparación y el desconocimiento de unicidad de soluciones. Este último sigue siendo un problema abierto a día de hoy. La existencia de soluciones se prueba a través de una aproximación del problema (PMEFP) en el que se regulariza, se elimina la degeneración y se restringe el dominio espacial. Para este problema aproximado se deducen unas ciertas estimaciones de energía que nos permiten pasar a límite mediante argumentos de compacidad. Un argumento esencial para la prueba de existencia (y posteriormente también para la velocidad de propagación) es la propiedad de decaimiento exponencial de la solución en el caso $m \in [2,3)$. Esta información es esencial para garantizar que las estimaciones de energía admiten los pasos al límite. Desafortunadamente, cuando $m \geq 3$, el decaimiento exponencial no es suficiente para asegurar compacidad debido que las estimaciones de energía degeneran. Los nuevos argumentos necesarios para el caso $m \geq 3$ están siendo investigados por los autores, y se espera que en no mucho tiempo aparezca un artículo con estos resultados. Es importante notar que para $m \geq 3$ los mismos argumentos para la prueba de velocidad de propagación finita son válidos una vez encontrado un procedimiento de construcción de soluciones aproximadas como en el caso $m \in (2,3)$.

Cuando $m \geq 2$, probamos la propiedad de velocidad de propagación finita para soluciones construidas del (PMEFP). Siendo más precisos, dado un dato inicial u_0 con soporte compacto, probamos que las soluciones construidas tienen también soporte compacto para cualquier $t > 0$. Para probar este resultado construimos de manera explícita una barrera $U(x,t)$ de soporte compacto en espacio para todo $t>0$ que está por encima de $u_0(x)$ a tiempo $t=0$. Argumentamos por contradicción al primer punto de contacto entre U y la solución construida u del problema (PMEFP).

La propiedad de velocidad de propagación infinita se prueba para el caso $m\in(1,2)$ en dimensión $N=1$. Las técnicas utilizadas rompen totalmente con lo descrito anteriormente. Se considera la función integrada $v(x,t)=\int_{-\infty}^{x}u(y,t)dy$ para la que se demuestra que cumple lo que llamamos el problema integrado $v_t=-|v_x|^{m-1}(-\Delta)^{1-s}v$. La ventaja de escribir el problema de esta manera es que para la ecuación integrada somos capaces de probar un principio de comparación parabólico adaptado al carácter no local de la ecuación. De esta manera, construimos de forma explícita una familia de sub-soluciones tales que para cualquier $x_0\in\mathbb{R}^N$ y cualquier $t_0>0$ existe un elemento de dicha familia Φ tal que $\Phi(x_0,t_0)>0$.

En el Capítulo 3 se muestran unos resultados que apuntan a que las soluciones del (PMEFP) para $m\in(1,2)$ y $N>1$ también tienen velocidad de propagación finita, pero en este caso el resultado es parcial.

0.2.3 Capítulo 3: Transformaciones de soluciones autosemejantes para ecuaciones fraccionarias del tipo medios porosos

Como continuación natural a los capítulos 1 y 2, en este capítulo presentamos una interesante relación entre las ecuaciones (FPME) y (PMEFP) y otros modelos fraccionarios del tipo medios porosos. Para ser más claros, consideramos las ecuaciones

$$u_t+(-\Delta)^s u^m=0, \tag{FPME}$$

$$v_t=\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\tilde{s}}v), \tag{PMEFP}$$

$$w_t=\nabla\cdot(w\nabla(-\Delta)^{-\widehat{s}}w^{\widehat{m}-1}), \tag{0.2.2}$$

así como la versión más general

$$z_t=\nabla(z^{\mathring{m}-1}\nabla(-\Delta)^{-\mathring{s}}z^{\mathring{n}-1}). \tag{0.2.3}$$

Damos unas fórmulas explícitas de transformaciones entre soluciones autosemejantes de estos cuatro modelos. En particular, las transformaciones que las relacionan con la ecuación (FPME) son de especial interés, ya que el comportamiento de esta ecuación es bastante conocido, y las soluciones autosemejantes dadas en [79] nos dan mucha información sobre el comportamiento de las soluciones de las otras ecuaciones. Por ejemplo entre (FPME) y (PMEFP) la transformación se tiene cuando $s=1-\tilde{s}$ en el rango de parámetros

$$m \in (1, \infty) \longleftrightarrow \tilde{m} \in (1, 2).$$

Esta relación es la que nos da otro resultado parcial de velocidad de propagación finita para el modelo (PMEFP) en dimensiones $N > 1$ cuando $\tilde{m} \in (1, 2)$. Este hecho complementa al resultado en dimensión $N = 1$ dado en el Capítulo 2. Otra interesante reflexión de esta relación entre soluciones es el hecho que el modelo (PMEFP) para $\tilde{m} = 2$ está directamente relacionado con el límite cuando $m \to +\infty$ del (FPME), llamado problema de la mesa (Ver [80] para más información y la Figura 1.14 para hacerse una idea del comportamiento de las soluciones).

0.2.4 Capítulo 4: Unicidad y propiedades de soluciones distribucionales de ecuaciones no locales generales del tipo medios porosos

Consideramos el siguiente problema de Cauchy:

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \quad \text{in} \quad Q_T := \mathbb{R}^N \times (0, T) \tag{VGL}$$

donde φ es una función continua y no decreciente y $\mathcal{L}^\mu$ es el operador no local general definido en la sección 0.1.3. La ecuación puede ser muy degenerada ya que φ podría ser constante a trozos y la medida μ podría no estar definida en alguna de las direcciones espaciales. Los resultados presentados en el Capítulo 4 forman parte del artículo [46] hecho en colaboración con Jørgen Endal y Espen R. Jakobsen de la Universidad Noruega de Ciencia y Tecnología.

Unicidad: Para un dato inicial $u_0 \in L^\infty(\mathbb{R}^N)$, probamos unicidad para una clase muy general de soluciones distribucionales de (VGL).

Estudio de la ecuación elíptica asociada: Una herramienta muy importante en la prueba de unicidad recae sobre las propiedades de la ecuación elíptica asociada

$$\epsilon v + \mathcal{L}^\mu[v] = g.$$

El propio Capítulo 4 provee una teoría general de existencia, unicidad y estimaciones para soluciones clásicas y de distribución para esta ecuación elíptica.

Existencia: Probamos existencia de soluciones distribucionales de (VGL) para datos iniciales $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Propiedades de la solución: Proveemos a las soluciones de una serie de propiedades tales como contracción en L^1, principio de comparación, decaimiento en norma L^1 y L^∞ y regularidad en tiempo. Además probamos conservación de masa bajo unas ciertas propiedades de regularidad de la función φ en el origen. Un ejemplo de que este tipo de condiciones de regularidad son necesarias se puede encontrar en [38], donde se prueba extinción en tiempo finito para el modelo (FPME) en un cierto rango del parámetro m.

Compacidad y convergencia: Probamos también dependencia continua en φ y μ. en particular, este resultado nos permite probar existencia de soluciones para cualquier ecuación de la forma

$$\partial_t u - L[u] = 0$$

donde L es cualquier operador elíptico del tipo $L^\sigma[\psi](x) := \mathrm{tr}(\sigma\sigma^T D^2\psi(x))$ para cualquier matriz $\sigma \in \mathbb{R}^{N\times N}$ (nótese que estos operadores pueden ser muy degenerados). En particular, esto generaliza el resultado de Brezis y Crandall en [19] donde la existencia es probada para $L = \Delta$.

Análisis numérico: Una de las más interesante novedades aparece aquí. La familia de medidas μ es tan general que, dado cualquier operador $\mathcal{L}^\mu$, somos capaces de encontrar un operador discreto $\mathcal{L}^\mu_D$ asociado que sigue estando en la clase de operadores considerados. Este hecho, junto con todos los resultados previos, nos provee de una aproximación semidiscreta con una teoría completa de existencia, unicidad, estabilidad y convergencia.

0.2.5 Capítulo 5: Investigación no publicada

Este último capítulo está dedicado a trabajos de investigación que por diferentes razones no se han publicado todavía, pero que de una manera u otra han sido muy relevantes en el desarollo de la tesis.

Aproximación numérica para la ecuación (FPME) general: En la Sección 5.1 se trata de encontrar un método numérico para la ecuación (FPME) con $s \in (0, 1)$ como extensión natural de los resultados del Capítulo 1 donde se cubría el caso $s = 1/2$. Proponemos un método numérico para el caso general cuando $s \in (0, 1)$ para el que probamos existencia, unicidad, principio del máximo y algunas propiedades sobre discretización del operador $(-\Delta)^s$. Desafortunadamente no hemos sido capaces de probar convergencia todavía, pero es un trabajo en progreso. La razón para no poder obtener convergencia de la misma manera que para el caso $s = 1/2$ es la pérdida de regularidad de la solución del problema de extensión en la nueva variable. Dentro de este capítulo se puede encontrar una discusión más profunda y detallada sobre estos hechos.

Método de extensión para el Laplaciano fraccionario inverso: Por otro lado, los resultados de la Sección 5.2 son motivados por una necesidad práctica aparecida durante el desarrollo del capítulo 2, donde se discute la velocidad de propagación de la ecuación (PMEFP) dependiendo del paramétro m. Cuando nos dimos cuenta de que para $m \in (1,2)$ la ecuación podría no tener velocidad de propagación finita vimos necesario hacer algún tipo de simulación numérica en la que pudiéramos observar si aparecían fronteras libres. El problema al que nos enfrentábamos es que los argumentos de extensión armónica para el Laplaciano fraccionario $(-\Delta)^s$ no existían para los Laplacianos fraccionarios inversos $(-\Delta)^{-s}$. En esta sección probamos que para los Laplacianos fraccionarios inversos también existe una teoría de extensión que convierte el problema no local en un problema local en una dimensión más. Como consecuencia de este resultado, presentamos un punto de partida para el tratamiento numérico local de la ecuación (PMEFP).

Artículos y colaboraciones

A continuación se enumeran los artículos a los que ha dado lugar la presente Tesis doctoral. Además, los métodos numéricos que aparecen en [Te14] y [TeVa13] y que han sido implementados por el autor, han sido utilizados en diversos artículos que se citan al final de esta sección.

Artículos publicados:

[Te14] Félix del Teso. Finite difference method for a fractional porous medium equation. *Calcolo 51* (2014), 615–638.

[StTeVa14] Diana Stan, Félix del Teso y Juan Luis Vázquez. Finite and infinite speed of propagation for porous medium equations with fractional pressure. *C. R. Math. Acad. Sci. Paris 352* (2014), 123–128.

[StTeVa15] Diana Stan, Félix del Teso y Juan Luis Vázquez. Transformations of self-similar solutions for porous medium equations of fractional type. *Nonlinear Anal. 119* (2015), 62–73.

Artículos aceptados:

[StTeVa15(2)] Diana Stan, Félix del Teso y Juan Luis Vázquez. Finite and infinite speed of propagation for porous medium equations with nonlocal pressure. *J. Differential Equations.* (2015) Preprint: arXiv:1506.04071v1 [math.AP]

Artículos enviados a publicación:

[EnJaTe15] Jørgen Endal, Espen R. Jakobsen y Félix del Teso. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. (2015) Preprint: arXiv:1507.04659v1 [math.AP]

Preprints:

[TeVa13] Félix del Teso y Juan Luis Vázquez. Finite difference method for a general fractional porous medium equation. (2013) Preprint: arXiv:1307.2474v1 [math.NA]

Colaboraciones:

- [68] Figura 3.4 y Figura 3.5
- [79] Figura 1.
- [80] Figura 1, Figura 2, Figura 3 y Figura 4.
- [83] Figura 1.

Introduction

The Heat Equation is the simplest and best known model describing diffusion processes,

$$\frac{\partial u}{\partial t} = \Delta u.$$

The function $u = u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ denotes the concentration or density of a certain substance which evolves in time on a domain. The description was originally given by Fourier in 1822 in his book "Théorie Analytique de la Chaleur". There exists also a deep relation between the Heat Equation and the Brownian motion of particles first shown in the pioneering paper of Einstein [44]. It is worthy to mention the discrete version of the Heat Equation, also known as random walk or drunk walk, which describes a linear diffusion process in a discrete set. In particular, under certain conditions on the discretization, the Heat Equation can be obtained as the limit of a random walk when the distance between the points of the discretization tends to zero.

Fortunately or unfortunately, not every physical phenomena can be described by a linear partial differential equation. For this reason, nonlinear partial differential equations have become a very important tool in the modern science and are used to describe physical systems like fluid mechanics, population dynamics, relativity, thermodynamics, etc. An important class of nonlinear partial differential equations are the parabolic equations of second order, frequently used to describe convection and diffusion process in the previously mentioned framework.

The study of nonlinear partial differential equations is quite complex. There is not a unified strategy to address the problems which naturally appears as can be the study of explicit solutions, existence, uniqueness and regularity of the solutions, asymptotic behavior, velocity of propagation and numerical methods. Most of this questions are studied in the present Thesis for a class of second order partial differential equations with nonlinear diffusion called Porous Medium Equations.

0.3 Motivation and physical applications

0.3.1 Physical model

The equation in which are based the models considered along this thesis was derived independently by Leibenzon and Muskat around 1930 while they where studying the flow of a gas through a porous medium. Some years before, in 1903, using a physical law that will lately inherit his name, Boussinesq had also considered this model for the study of groundwater infiltrations.

Consider a continuum media which can be either a liquid or a big enough population and denote by $u(x,t) > 0$ to the density of distribution that evolves in time following a velocity field $v(x,t)$ which can also vary with the time. The corresponding continuity equation is formulated as

$$\frac{\partial u}{\partial t} + \nabla \cdot (u \cdot v) = 0.$$

Two physical assumptions are made now:

(a) The velocity of fluids in a porous media behave according to Darcy's law, that is, v is a potential of the form
$$v = -\nabla p,$$
where p denotes the pressure.

(b) There exists a relation between the pressure p and the density of the fluid u: Leibenzon and Muskat obtained a relation of the form
$$p = f(u),$$
where f is a nondecreasing scalar function. Such function $f(u)$ is linear when the flow is isothermal while is a higher power of the form $f(u) = cu^{m-1}$ with $c > 0$ and $m > 1$ when the flow is adiabatic.

The linear dependence of f with respect to u was also obtained by Boussinesq and it was used in the equation $u_t = c\nabla \cdot (u\nabla u) = \frac{c}{2}\Delta u^2$, used for modeling water infiltration in an almost horizontal soil layer.

Up to constants, the classical Porous Medium Equation can be written in the form $u_t = \nabla \cdot (u^{m-1}\nabla u)$, or in its best known form

$$\frac{\partial u}{\partial t} = \Delta u^m.$$

This typical model of nonlinear equation has been investigated by many authors. We refer to the book [77] of Juan Luis Vázquez for a very complete reference about the properties of this equation.

0.3.2 Nonlinear diffusion

As mentioned, one of the most popular models of nonlinear diffusion is the Porous Medium Equation,

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{for} \qquad m > 1. \tag{PME}$$

The (PME) belongs to the more general class of parabolic equations

$$\frac{\partial u}{\partial t} = \nabla \cdot (A(u)\nabla u),$$

where $A(u)$ is usually called the diffusion coefficient. In the case of the (PME) we have that $A(u) = u^{m-1}$, which makes the equation to be degenerated for $m > 1$. Equation (PME) is also studied when $m < 1$ and then it becomes singular. This last case is called Fast Diffusion Equation. John R. King considered the Fast Diffusion Equation in [61] while he was investigating the diffusion of impurities in silicon.

The terminology *slow/fast diffusion* refers to the property of finite and infinite speed of propagation of the solutions. This means that when $m > 1$, if the initial data is compactly supported, the solutions of (PME) have compact support for all times $t > 0$. In particular, this fact open the gates to the so-called *free boundaries*, where the properties of the boundaries of the domain where the solutions are nonzero are studied. On the other hand, when $m \leq 1$ the solutions of the (PME) have infinite speed of propagation, that is, given a nonnegative initial data (that could possibly be compactly supported), the solution automatically becomes strictly positive in the whole space for any time $t > 0$.

0.3.3 Nonlocal diffusion

In the diffusion processes considered above, the effect of interaction between particles is local. In general, many of these interactions are not merely local and the use of more general operators than the Laplacian is required. The best known nonlocal operator is the so-called Fractional Laplacian (Stein [73], Landkof [63]) which has a lot of equivalent definitions. Let us denote the Fourier Transform by $\mathcal{F}$. We define the fractional Laplacian $(-\Delta)^s$ when $s \in (0,1)$ for functions f in the Schwartz class as

$$\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi).$$

This definition generalizes in some sense the well known property $\mathcal{F}(\Delta f)(\xi) = -|\xi|^2\mathcal{F}(f)(\xi)$ of the standard Laplacian. In an equivalent way, the fractional Laplacian operator can also be defined through the following singular integral,

$$(-\Delta)^s f(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+2s}}dy,$$

where P.V denotes the principal value of the integral and $C_{N,s} = \pi^{-(2s+N/2)}\frac{\Gamma(\frac{N}{2}+s)}{\Gamma(-s)}$ is a normalization constant. Formally, if we make $s \to 1^-$ we recover the local Laplacian while for $s \to 0^+$ we get the identity. We refer to [75] for a very illustrative proof of the equivalence of both definitions. There exists a third equivalent way of defining the fractional Laplacian. It is given through the heat semigroup associated to the operator $-\Delta$:

$$(-\Delta)^s f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t(-\Delta)}f(x) - f(x)\right)\frac{dt}{t^{1+s}}.$$

We refer to the article [74] by Stinga and Torrea for a further discussion on this definition. The inverse operator of the fractional Laplacian can be formulated as a convolution with the Riesz potential, and then for $N > 2s$ we have the explicit formula

$$(-\Delta)^{-s} f(x) = C_{N,-s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}dy.$$

In probability, the fractional Laplacian is the infinitesimal generator of an $(s/2)$-stable process, also called Lévy stable processes. It is important for us to note that the fractional Laplacian belongs to a much more general class of nonlocal diffusion operators that are also generators of symmetrical Lévy processes. Let μ be a positive and symmetric Radon measure (possibly singular) which satisfies $\int_{\mathbb{R}^N} \min\{|z|^2, 1\}\, \mathrm{d}\mu(z) < +\infty$. We define for $\psi \in C_c^\infty(\mathbb{R}^N)$ the generator of a symmetrical nonlocal diffusion Lévy process as

$$\mathcal{L}^\mu[\psi](x) = \int_{\mathbb{R}^N\setminus\{0\}} \psi(x+z) - \psi(x) - z\cdot D\psi(x)\mathbf{1}_{|z|\le 1}\, \mathrm{d}\mu(z),$$

where $\mathbf{1}_{|z|\le 1}$ denotes the indicator function in the set $\{|z| \le 1\}$. Note that for $\mathrm{d}\mu(z) = \frac{C_{N,s}}{|z|^{N+2s}}dz$, we have $\mathcal{L}^\mu = (-\Delta)^s$.

The anomalous diffusion processes are frequently described by nonlinear and nonlocal modes. We present now the three nonlocal diffusion models of porous medium type that will be the main topic of study in this thesis.

(I) Fractional Porous Medium Equation. For $s \in (0,1)$ and $m \geq 0$ we consider the equation

$$\frac{\partial u}{\partial t} + (-\Delta)^s u^m = 0. \tag{FPME}$$

This model is a possible generalization to the nonlocal framework of the classical (PME) (they are in fact the same for $s = 1$). There exists a lot of differences in the behaviors of the solutions of the (PME) and the (FPME). The first and more surprising one is the fact that the solutions of the (FPME) have infinite speed of propagation, while the solutions of the (PME) have finite speed of propagation. This property of the (FPME) is a consequence of the nonlocal character of the operator $(-\Delta)^s$ when $s \in (0,1)$. The first known research works about the (FPME) appears in the series of articles [37–39, 82] by Arturo de Pablo, Fernando Quirós, Ana Rodríguez and Juan Luis Vázquez where the equation is considered in $\mathbb{R}^N$. Later, Matteo Bonforte, Yannick Sire and Juan Luis Vázquez deal in [16, 18] with the equation posed in bounded domains. Chapter 1 y part of Chapter 5 of the present manuscript are devoted to the numerical study of the equation (FPME).

(II) Porous Medium Equation with Fractional Pressure. We consider for $s \in (0,1)$ the Porous Medium Equation written in divergence form and assume that the pressure p is related to the density u through a nonlocal operator like the inverse fractional Laplacian, that is,

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(u \nabla (-\Delta)^{-s} u\right). \tag{CV}$$

This model was considered by Luis Caffarelli and Juan Luis Vázquez in [25], where they show existence of solutions and the property of finite speed of propagation of them. Later, the same authors in collaboration with Fernando Soria study the regularity of the solutions in [22, 23].

One of the main topics in the research presented in this Ph.D. Thesis is based on the study of a generalization of the model (CV) for more general fluxes. In Chapter 2 we study in a very precise way the velocity of propagation of the solutions of the model

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(u^{m-1} \nabla (-\Delta)^{-s} u\right). \tag{PMEFP}$$

Moreover, in Chapter 3 we show a number of relations between the solutions of the models (FPME) and (PMEFP). Even more, in the last part of Chapter 5 we propose a possible starting point for the numerical study of the (PMEFP) through a generalization of the extension problems introduced by Luis Caffarelli and Luis Silvestre in [21] adapted to the inverse fractional Laplacians.

(III) Generalized Nonlocal Porous Medium Equation. A more general nonlinear and nonlocal diffusion models than (FPME) can be considered. In Chapter 4 we study the distributional solutions of the problem

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0, \tag{VGL}$$

where $\mathcal{L}^\mu$ is the general nonlocal operator defined in Section 0.1.3 and the nonlinearity φ is any continuous nondecreasing scalar function. Viscosity and entropy solutions for this kind of equations have been previously considered by authors like Espen R. Jakobsen, Jørgen Endal, Simone Cifani or Kenneth H. Karlsen in [32, 45, 58].

0.4 Main results

We present now the main results obtained during the elaboration of the present Ph.D. Tesis

0.4.1 Chapter 1: Numerical analysis for the Fractional Porous Medium Equation

During the preparation of the articules previously cited about the Fractional Porous Medium Equation,

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{1}{2}} u^m = 0 \qquad x \in \mathbb{R}^N,\ t > 0 \tag{FPME}$$

for $m \geq 1$, the authors realized that, at that time, all the existing numerical approximations for this equation were based on the study of quadrature methods for singular integrals. A completely new direction was proposed then based on an idea of Caffarelli an Silvestre. In this chapter we propose an explicit finite difference numerical scheme for the (FPME). The main novelty of this scheme relays on the local treatment of the nonlocal equation (FPME): We present a finite difference method for the following evolution problem in $N + 1$ dimensions,

$$\begin{cases} \Delta w = 0, & x \in \mathbb{R}^N,\ y > 0,\ t > 0, \\ \dfrac{\partial w^{1/m}}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y}(x,0,t), & x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ w(x,0,0) = f^m(x), & x \in \mathbb{R}^N. \end{cases} \tag{0.4.1}$$

Problems (FPME) and (0.4.1) are equivalent in the sense that $u(x,t)$ is the trace (when $y = 0$) of $w^{1/m}(x,y,t)$. We prove existence, uniqueness and a maximum principle for the solutions of the numerical method. Moreover, we prove that the numerical solutions converge to the solutions of (FPME) with a rate of order $O(\Delta t + \Delta x)$ where Δt and Δx are the distances between points of the time and space discretization respectively. To finish, we present some numerical experiments of different kinds where we observe that the proved order of convergence is the correct one.

Contemporaneously and also later to these results about numerical methods for fractional equations, other authors have found this field interesting and they have developed other numerical approximations, some of them in the same direction but some others in a completely different way. To cite some of them, Chen, Nochetto, Otárola and Salgado present in [29] a method based in the harmonic extension (0.4.1), but using finite elements instead of finite differences. On the other hand Cifani and Jakobsen in [33] deal directly with the nonlocal formulation given by the singular integral. Huang and Oberman combine in [54] both finite difference and numerical quadrature for the integrals. Similar ideas were used by Biswas, Jakobsen and Karlsen for Bellman equations some years before in [14] . Recently, Ciaurri, Roncal, Stinga, Torrea and Varona present in [30] a completely different idea to the ones mentioned above; their numerical method is based in approximating the fractional Laplacian by fractional powers of the discrete Laplacian. For that purpose, they start with the definition of the fractional Laplacian given by the heat Semigroup given in section 0.3.3.

0.4.2 Chapter 2: Porous Medium Equation with fractional pressure

This chapter is devoted to the study of the following diffusion equations in divergence form

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot \left(u^{m-1}\nabla(-\Delta)^{-s}u\right) \qquad x \in \mathbb{R}^N,\ t > 0, \tag{PMEFP}$$

for $m > 1$, $s \in (0,1)$ and $u(x,t) \geq 0$ in dimension $N \geq 1$. We consider the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N$$

where $u_0 : \mathbb{R}^N \to [0,\infty)$ is a bounded function (we will sometimes assume that it is also compactly supported or that it has some kind of rapid decay at infinity). As we mentioned before, in this model, written in divergence form, the pressure p is related to the density u through a fractional operator. To be more precise, $p = (-\Delta)^{-s}$ for $s \in (0,1)$. The particular case of (PMEFP) when $m = 2$ was introduced by Luis Caffarelli and Juan Luis Vázquez in [25] as a nonlinear diffusion model of porous medium type with nonlocal diffusion effects.

The results of this chapter have appeared in the articles [69, 70] in collaboration with Juan Luis Vázquez and Diana Stan. We prove an existence result for a class of weak solutions in the range $m \in (1, 3)$. These weak solutions are the result of an approximation method by regular solutions. We will refer to them as *constructed weak solutions*.

Motivated by the results proved in [25], the main goal of this research was to understand the velocity of propagation for the solutions of (PMEFP). They established that the constructed weak solutions of for (PMEFP) in the case $m = 2$ have finite speed of propagation. In this way, we discover that the case $m = 2$ was a critical point in the velocity of propagation: under certain hypothesis we get that when $m \in [2, \infty)$ the solutions have **finite speed of propagation**, while for $m \in [1, 2)$ they have **infinite speed of propagation**.

The main problems that we had to deal with were the lack of a comparison principle for solutions and ignorance of a result of uniqueness. This last one still being an open problem nowadays. We prove the existence of solutions through an approximation of problem (PMEFP) where we regularize, eliminate the degeneracy and restrict the spatial domain. We establish some energy estimates that give us the required compactness for the approximated problem that allow us to pass to the limit. An essential tool in the proof of existence (and later also for the velocity of propagation) in the case $m \in [2, 3)$ is the property of exponential decay of the solutions. This information is crucial to ensure that the energy estimates pass to the limit in a proper way. Unfortunately, when $m \geq 3$ the exponential decay of the solutions is not enough to ensure the required compactness since the energy estimates become degenerate. New compactness arguments are being studied by the authors in the case $m \geq 3$ and we expect to publish soon an article about this fact. It is important to note that for $m \geq 3$ the same arguments for the proof of finite speed of propagation are valid once we provide an approximation argument as in the case $m \in (2, 3)$.

For $m \geq 2$ we prove the property of finite speed of propagation for constructed weak solutions of (PMEFP). To be more precise, given a compactly supported initial data u_0, we prove that the constructed weak solutions have also compact support for every time $t > 0$. To prove this result, we construct an explicit barrier $U(x, t)$ with compact support for all $t > 0$ which is above $u_0(x)$ at time $t = 0$. We argue by contradiction at the first contact point between U and the solution u of the problem (PMEFP).

The property of infinite speed of propagation is proved in the case $m \in (1,2)$ for dimension $N = 1$. The techniques used here are completely different with the ones described above. We consider the integrated function $v(x,t) = \int_{-\infty}^{x} u(y,t)dy$ for which we prove that follows what we call the integrated problem $v_t = -|v_x|^{m-1}(-\Delta)^{1-s}v$. The main advantage of writing the problem in this way is that we are able to prove a parabolic comparison principle adapted to nonlocal character of the integrated problem. In this way, we construct an explicit family of sub-solutions such that for every $x_0 \in \mathbb{R}^N$ and every $t_0 > 0$ there exists an element of the family Φ such that $\Phi(x_0, t_0) > 0$.

In Chapter 3 we show some results which point out that the solutions of the (PMEFP) for $m \in (1,2)$ and $N > 1$ also have infinite speed of propagation, but in this case the result is partial in the sense that it is not valid for every constructed weak solution.

0.4.3 Chapter 3: Transformations of self-similar solutions for fractional equations of porous medium type

As a natural consequence of chapters 1 and 2, we present in this chapter an interesting and useful relation between equations (FPME) and (PMEFP) and some fractional models of porous medium type. Te be more specific, we consider the following equations:

$$u_t + (-\Delta)^s u^m = 0, \tag{FPME}$$

$$v_t = \nabla \cdot (v^{\widetilde{m}-1} \nabla (-\Delta)^{-\widetilde{s}} v), \tag{PMEFP}$$

$$w_t = \nabla \cdot (w \nabla (-\Delta)^{-\widehat{s}} w^{\widehat{m}-1}), \tag{0.4.2}$$

and the more general version

$$z_t = \nabla (z^{\mathring{m}-1} \nabla (-\Delta)^{-\mathring{s}} z^{\mathring{n}-1}). \tag{0.4.3}$$

We give explicit formulas of transformations between self-similar solutions of these four models. In particular, the transformations which relate with equation (FPME) are of special interest because the behavior of this equation is petty well know and the semi-explicit self-similar solutions given in [79] give us a lot of information about the behavior of the other equations. For example, the transformation between (FPME) and (PMEFP) when $s = 1 - \tilde{s}$ is given in the range of the parameters

$$m \in (1, \infty) \longleftrightarrow \tilde{m} \in (1,2).$$

This relations give us another partial result in the speed of propagation for model (PMEFP) in dimensions $N > 1$ for $m \in (1,2)$. This fact complements the result in

$N = 1$ given in Chapter 2. Another interesting consideration of this relation between solution is the fact that model (PMEFP) for $\tilde{m} = 2$ is directly related to the limit as $m \to +\infty$ of equation (FPME), called the Mesa problem (See [80] for more information and Figure 1.14 for an idea of how the solutions of (FPME) behave for a big m).

0.4.4 Chapter 4: Uniqueness and properties of distributional solutions of general nonlocal equations of porous medium type

We consider the following Cauchy problem:

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \quad \text{in} \quad Q_T := \mathbb{R}^N \times (0,T) \tag{VGL}$$

where φ is continuous and nondecreasing and $\mathcal{L}^\mu$ is a very general nonlocal Lévy operator defined in section 0.3.3. Equation (VGL) can be very degenerate since φ could be somewhere constant, and the measure μ will possibly be not defined in some spatial directions. The results presented in Chapter 4 are part of the article [46] done in collaboration with Jørgen Endal and Espen R. Jakobsen from the Norwegian University of Science and Technology.

Uniqueness: For an initial data $u_0 \in L^\infty(\mathbb{R}^N)$, we prove uniqueness for a very general class of distributional solutions of (VGL).

Study of the associated elliptic equation: A key tool in the proof of Uniqueness of (VGL) relies on the properties of the solution of the elliptic problem

$$\epsilon v + \mathcal{L}^\mu[v] = g.$$

A general theory of existence, uniqueness and estimates for both classical and distributional solutions is provided.

Existence: We show existence of distributional solutions of (VGL) for initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Properties of the solution: We provide a series of properties for the solution of this general model such as L^1-contraction, comparison principle, L^1 and L^∞ decay and regularity in time. Moreover, mass conservation is also proved provided a condition on the smoothness of φ close to the origin. The Fast Diffusion Equation is a well known example with lack of conservation of mass where this kind of regularity on φ is not assumed. This equation is well studied in [38] in the fractional case.

Compactness and convergence: Continuous dependence on φ, μ is showed. In particular, this result allows us to prove existence for any equation of the form

$$\partial_t u - L[u] = 0$$

where L is any local elliptic operator of the form $L^\sigma[\psi](x) := \mathrm{tr}(\sigma\sigma^T D^2\psi(x))$ for any matrix $\sigma \in \mathbb{R}^{N\times N}$ (note that this operator can be strongly degenerate). This generalizes the work by H. Brezis and M. G. Crandral in [19] where the existence result was proved for $L = \Delta$.

Numerical analysis: One of the main novelties of the present work appears here. The family of measures μ considered are such general that given any operator $\mathcal{L}^\mu$, we find a discrete operator $\mathcal{L}^\mu_D$ which still belonging to the class of operators considered. This fact, together with all the previous results, provide a semi-discrete approximation with a full theory of existence, uniqueness, stability and convergence.

0.4.5 Chapter 5: Non-published research

This last chapter is devoted to some research works done by the author that for different reasons have not been published yet, but in a way or another they have been very relevant part during the preparation of this Thesis.

Numerical approximation of the general equation (FPME): In Section 5.1 we try to find a numerical method for equation (FPME) with $s \in (0,1)$ as the natural extension of the results given in Chapter 1 where the case $s = 1/2$ was covered. We propose a numerical scheme general case when $s \in (0,1)$ for which we prove existence, uniqueness, a maximum principle and some properties about the discretization of the operator $(-\Delta)^s$. Unfortunately, we have not been able to prove convergence yet, but is still a work in progress. The reason why convergence can not be obtained in the same way as in the case when $s = 1/2$ is the lack of regularity of the solution of the extension problem in the new variable. A further discussion of this fact can also be found in this chapter.

Extension method for the inverse fractional Laplacian: On the other hand, results given in 5.2 are motivated for the practical necessity that appeared during the preparation of Chapter 2, where the speed of propagation of model (PMEFP) is discussed depending on the parameter m. When we realized that for $m \in (1,2)$ the equation could not have finite speed of propagation, we had the necessity of some kind of numerical simulation of the solution that could allow us to observe the existence of free boundaries. In that section we show an extension method for the inverse fractional Laplacian $(-\Delta)^{-s}$

which turns the nonlocal problem into a local problem in one more dimension. As a consequence of this result, we present a possible starting point for local numerical treatment of equation (PMEFP).

Articles and collaborations

The following articles have been written as consequence of the present Ph.D. Thesis. Moreover, numerical methods shown in [Te14] and [TeVa13] which have been implemented by the author, have been used in several articles and we cite them at the end of this section.

Published articles:

[Te14] `Félix del Teso`. Finite difference method for a fractional porous medium equation. *Calcolo 51* (2014), 615–638.

[StTeVa14] `Diana Stan, Félix del Teso and Juan Luis Vázquez`. Finite and infinite speed of propagation for porous medium equations with fractional pressure. *C. R. Math. Acad. Sci. Paris 352* (2014), 123–128.

[StTeVa15] `Diana Stan, Félix del Teso and Juan Luis Vázquez`. Transformations of self-similar solutions for porous medium equations of fractional type. *Nonlinear Anal. 119* (2015), 62–73.

Accepted articles:

[StTeVa15(2)] `Diana Stan, Félix del Teso and Juan Luis Vázquez`. Finite and infinite speed of propagation for porous medium equations with nonlocal pressure. *J. Differential Equations.* (2015) Preprint: arXiv:1506.04071v1 [math.AP]

Articles submitted for publication:

[EnJaTe15] `Jørgen Endal, Espen R. Jakobsen and Félix del Teso`. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. (2015) Preprint: arXiv:1507.04659v1 [math.AP]

Preprints:

[TeVa13] `Félix del Teso and Juan Luis Vázquez`. Finite difference method for a general fractional porous medium equation. (2013) Preprint: arXiv:1307.2474v1 [math.NA]

Collaborations:

- [68] Figure 3.4 and Figure 3.5
- [79] Figure 1.
- [80] Figure 1, Figure 2, Figure 3 and Figure 4.
- [83] Figure 1.

Chapter 1

Finite difference method for a fractional porous medium equation

1.1 Introduction

This chapter is concerned with a numerical method for the Cauchy problem,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{1/2}(|u|^{m-1}u) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1.1}$$

for exponents $m \geq 1$ and space dimension $N \geq 1$. We also prove existence and uniqueness of solutions for the numerical scheme. Moreover, convergence to the classical solution of (1.1.1) is also proved via a maximum principle. We extend the results for equations with a more general nonlinearity instead of $|u|^{m-1}u$. We will consider nondecreasing $\varphi(u)$ with good regularity conditions that will be specified later. The general theory of existence, uniqueness and regularity of solutions for the problem (1.1.1) has been studied by A. de Pablo, F. Quirós, A. Rodríguez and J.L. Vázquez in [37]. They also study a more general case with $(-\Delta)^{\sigma/2}$ for $\sigma \in (0,2)$ in [38]. Even more, in [39] they consider a logarithmic diffusion $\varphi(u) = \log(u+1)$ as de natural limit as $m \to 0$ of $\varphi(u) = u^m$. Higher order regularity for solutions of (1.1.1) are proved in [82].

We recall that the nonlocal operator $(-\Delta)^{1/2}$ is well defined via Fourier transform for any function f in the Schwartz class as the operator such that

$$\mathcal{F}((-\Delta)^{1/2}u)(\xi) = |\xi|\mathcal{F}(u)(\xi).$$

Moreover, it can also be defined for a more general class of functions with the singular integral representation

$$(-\Delta)^{1/2}f(x) = C_N \text{P.V.} \int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+1}}dy,$$

where $C_N = \pi^{-\frac{N+1}{2}}\Gamma(\frac{N+1}{2})$ is a normalization constant. For an equivalence of both formulations see for example [75]. We refer to [63] for the typical arguments of potential theory related to fractional laplacian.

Previous works in numerical analysis for nonlocal equations of this type are done by Cifani, Jakobsen, and Karlsen in [32–34]. In particular they formulate some convergent numerical scheme for entropy and viscosity solutions. The main difference of the present work is that we don not deal directly with the integral formulation of the fractional laplacian, instead of this, we pass through the Caffarelli-Silvestre extension method (see [21]) with implies solving a problem involving just local operators in one more space dimension. The numerical analysis of the elliptic PDE $(-\Delta)^{\sigma/2}u = f$ for $\sigma \in (0,2)$ in a bounded domain Ω with zero boundary data has been recently studied by Chen, Nochetto, Otárola and Salgado in [29] using finite elements in the extension problem.

1.2 Local formulation of the non-local problem

1.2.1 The problem in $\mathbb{R}^N$

Our aim is to find numerical approximations for positive and smooth solutions of the following porous medium equation with fractional diffusion,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^{1/2}u^m(x,t) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2.1}$$

with $m \geq 1$ and nonnegative initial data $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The general theory for existence, uniqueness and regularity of solutions for problem (1.2.1) can be found in [37, 82]. In particular, they state that problem (1.2.1) is equivalent to the so-called extension formulation,

$$\begin{cases} \Delta_{x,y} w(x,y,t) = 0, & x \in \mathbb{R}^N,\ y > 0,\ t > 0, \\ \dfrac{\partial w^{1/m}}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y}(x,0,t), & x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ w(x,0,0) = f^m(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2.2}$$

The equivalence between (1.2.1) and (1.2.2) holds in the sense of trace and harmonic extension, that is $u(x,t) = Tr(w^{1/m}(x,y,t))$ and $w(x,y,t) = E(u^m(x,t))$. In (1.2.2), $\Delta_{x,y}$ denotes the $N+1$ dimensional laplacian operator,

$$\Delta_{x,y} = \sum_{l=1}^{N} \frac{\partial^2}{\partial x_l^2} + \frac{\partial^2}{\partial y^2}.$$

1.2.2 The problem in a bounded domain

We are going to construct numerical approximations for the solution of (1.2.2) posed in the whole $\mathbb{R}_+^{N+1}$ as a monotone sequence of numerical solutions of the problem posed in a bounded domain. Consider the following positive constants $\{X_l\}_{l=1}^N, Y, T \in \mathbb{R}_+$ and define the bounded domain $\Omega \subset \mathbb{R}_+^{N+1}$ as

$$\Omega = (-X_1, X_1) \times \ldots \times (-X_N, X_N) \times (0, Y) \quad \text{with} \quad \Gamma = \partial\Omega.$$

For convenience on the kind of boundary conditions considered on different regions of the boundary, we divide Γ the in two parts (see Figure 1.1 for a picture showing the one dimensional case),

$$\Gamma_d = [-X_1, X_1] \times \ldots \times [-X_N, X_N] \times \{0\}, \qquad \Gamma_h = \Gamma \backslash \Gamma_d.$$

According to the notation introduced above, we formulate the following problem posed in the bounded domain Ω,

$$\begin{cases} \Delta w(x,y,t) = 0, & (x,y) \in \Omega,\ t \in (0,T], \\ \dfrac{\partial w^{1/m}}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y}(x,0,t), & (x,y) \in \Gamma_d,\ t \in (0,T], \\ w(x,0,0) = f^m(x), & (x,y) \in \Gamma_d, \\ w(x,y,t) = 0, & (x,y) \in \Gamma_h. \end{cases} \tag{1.2.3}$$

Note that a new boundary condition has appear. This is due to the fact that we have introduced an artificial boundary Γ_h. Homogeneous Dirichlet boundary conditions are

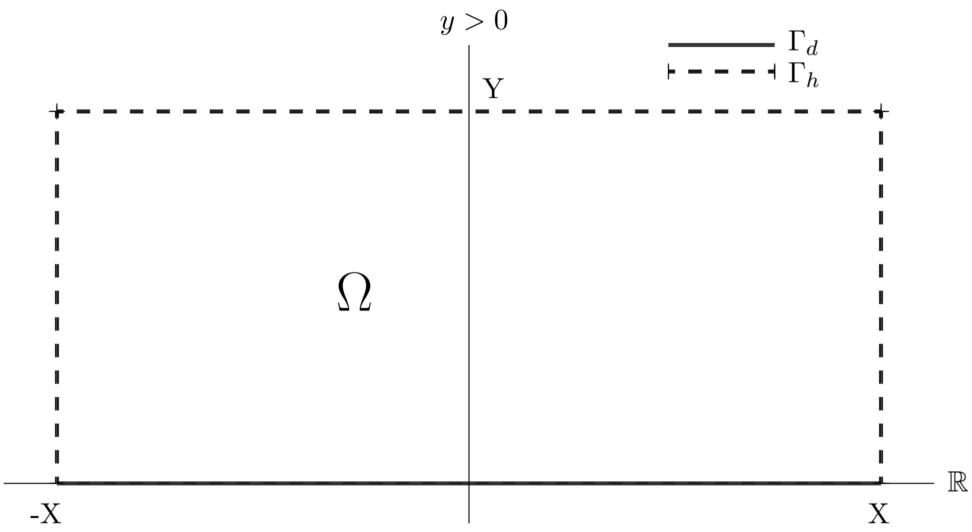

FIGURE 1.1: Bounded domain Ω

imposed on Γ_h in order to have a monotone approximations to the problem posed in $\mathbb{R}^{N+1}$.

In what follows, we will consider problem (1.2.3) for dimension $N = 1$ in order to simplify de notation. All the arguments that appear along this chapter are also valid for $N > 1$ and can be adapted with a multiindex notation without any extra effort (See subsection 1.7.1 for more details).

1.3 The numerical method

1.3.1 Discrete formulation

In order to find a numerical approximation for the solution of Problem (1.2.3) when $(x, y) \in \Omega$ and $t \in [0, T]$, we need define a space and time discretization. We do it as follows:

Time discretization: We define a discretization of the time interval $[0, T]$ by choosing the number of steps J. Then, the nodes of the mesh are $t_j = j\Delta t$ for $j = 0, \ldots, J$ where $\Delta t = T/J$ is the separation between them.

Space discretization: We define now a discretization of the domain $\overline{\Omega} = [-X, X] \times [0, Y]$. Let I be the number of steps in $[-X, X]$. Then, the nodes of the mesh are $x_i = i\Delta x - X$ for $i = 0, \ldots, I$ where $\Delta x = 2X/I$. In the same way, if K is the number of steps in $[0, Y]$, the nodes of the mesh are $y_k = k\Delta y$ for $k = 0, \ldots, K$ where $\Delta y = Y/K$. See Figure 1.2 for a representation of this discretization

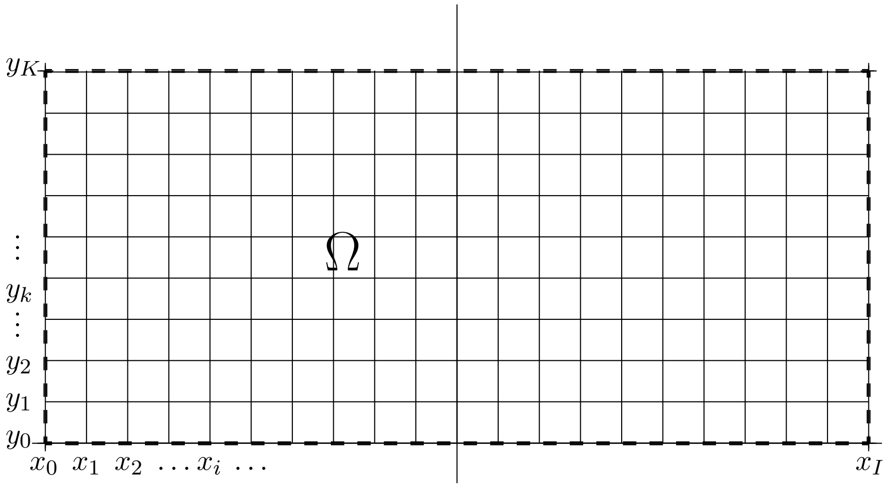

FIGURE 1.2: Discretization of $\overline{\Omega}$ with nodes

We will use the notation

$$(w_j)_i^k := w(x_i, y_k, t_j), \tag{1.3.1}$$

for the values of the solution w to Problem (1.2.3) in the points of the mesh. Moreover we will denote by $(W_j)_i^k$ to values of the numerical solution in the points of the mesh, that is,

$$(W_j)_i^k \approx w(x_i, y_k, t_j). \tag{1.3.2}$$

1.3.2 Numerical scheme

From now on, let us assume that $\Delta y = \Delta x$. For each time step $j = 1, \dots, J$ the numerical approximation is the solution to the following linear system of equations

$$\begin{cases} \dfrac{(W_j)_{i+1}^k + (W_j)_{i-1}^k + (W_j)_i^{k+1} + (W_j)_i^{k-1} - 4(W_j)_i^k}{\Delta x^2} = 0, & 0 < i < I, 0 < k < K, \\ (W_j)_i^0 = \left(\dfrac{\Delta t}{\Delta x}((W_{j-1})_i^1 - (W_{j-1})_i^0) + [(W_{j-1})_i^0]^{1/m}\right)^m, & \text{if } 0 < i < I, \\ (W_j)_i^k = 0, & \text{otherwise.} \end{cases} \tag{1.3.3}$$

Note that $(W_j)_i^0$ for $i = 1, \dots, X$ are the only variables involved in the second equation of (1.3.3) for a fixed time j. The other terms are just the values of the numerical solution in the previous time step. In other words, the second equation of (1.3.3) is discrete Dirichlet boundary condition on Γ_d. Due to this fact, we need a numerical approximation $(W_0)_i^j$ at time $t_0 = 0$ in order to have a solution of the linear system of equations (1.3.3) for each $j = 1, \dots, J$. We use the solution of the of the following linear system of equation to for that purpose

$$\begin{cases} \dfrac{(W_0)_{i+1}^k + (W_0)_{i-1}^k + (W_0)_i^{k+1} + (W_0)_i^{k-1} - 4(W_0)_i^k}{\Delta x^2} = 0, & 0 < i < I, 0 < k < K, \\ (W_0)_i^0 = f^m(x_i), & \text{if } 0 < i < I, \\ (W_0)_i^k = 0, & \text{otherwise.} \end{cases} \tag{1.3.4}$$

1.3.3 Local truncation error

The precise definition of the local truncation error can be found in the proof of Theorem 1.3.1, but essentially the local truncation error $(\tau_j)_i^k$ at each point (x_i, y_k, t_j) of the mesh is the error that comes from plugging the solution w to Problem (1.2.3) into the numerical numerical scheme (1.3.3). We will also use the notation

$$\Lambda = \max_{i,k,j} |(\tau_j)_i^k|. \tag{1.3.5}$$

$$\max_{i,j}\{|(\tau_j)_i^0|\} \tag{1.3.6}$$

Theorem 1.3.1 (Local truncation error)**.** *Let w be the solution to Problem* (1.2.3) *for $m \geq 1$. For every j=0,...,J we have that*

(a) (Interior nodes) If $0 < i < I$ and $0 < k < K$ then

$$|(\tau_j)_i^k| \leq O(\Delta x^2).$$

(b) (Γ_d nodes) If $0 < i < I$ and $k = 0$, then

$$|(\tau_j)_i^k| = O(\Delta t(\Delta t + \Delta x)).$$

Moreover,

$$\Lambda = O(\Delta t(\Delta x + \Delta t) + \Delta x^2). \tag{1.3.7}$$

Proof. The local truncation error in the nodes situated on the artificial boundary Γ_h is zero. This is a direct consequence of the choice $(W_j)_i^k = 0$ in the nodes $(x_i, y_k) \in \Gamma_h$ in the same way as $w(x, y, t) = 0$ if $(x, y) \in \Gamma_h$ in Problem (1.2.3).

Interior nodes: If $0 < i < I$ and $0 < k < K$, then

$$\begin{aligned}(\tau_{j-1})_i^k &:= \frac{1}{\Delta x^2}\left[(w_{j-1})_{i+1}^k + (w_{j-1})_{i-1}^k + (w_{j-1})_i^{k+1} + (w_{j-1})_i^{k-1} - 4(w_{j-1})_i^k\right] \\ &= \Delta w(x_i, y_k, t_{j-1}) + O(\Delta x^2) = O(\Delta x^2).\end{aligned}$$

where the last identity is just a consequence of the first equation of (1.2.3).

Γ_d **nodes:** If $0 < i < I$ and $k = 0$, then

$$\begin{aligned}(\tau_{j-1})_i^0 &:= \frac{\Delta t}{\Delta x}\left[(w_{j-1})_i^1 - (w_{j-1})_i^0\right] + [(w_{j-1})_i^0]^{1/m} - [(w_j)_i^0]^{1/m} \\ &= \Delta t\left[\frac{\partial w}{\partial y}(x_i, 0, t_{j-1}) + O(\Delta x)\right] - \Delta t\left[\frac{\partial w^{1/m}}{\partial t}(x_i, 0, t_{j-1}) + O(\Delta t)\right] \\ &= O(\Delta t \Delta x) + O(\Delta t^2) = O(\Delta t(\Delta t + \Delta x)).\end{aligned}$$

where the last identity comes from the second equation of (1.2.3). The previous computations are done assuming that the solution w of (1.2.3) is regular enough. This kind of high order regularity results can be found in [82]. □

1.3.4 Existence and uniqueness of the numerical solution

We define the following quantity needed for the results coming from now:

$$b_{\max} = \max_{x \in \mathbb{R}} \{f^m(x)\}. \tag{1.3.8}$$

Moreover, for $m \geq 1$, the function $\varphi(s) = s^m$ is such that $\varphi'(s) = m s^{m-1}$ is a non-decreasing and locally bounded function and it is well defined for all $s \geq 0$.

Theorem 1.3.2 (Discrete maximum principle). *Let $(W_j)_i^k$ be a solution to Problem (1.3.3) with $m \geq 1$. Assume*

$$\Delta t \leq C(m, f)\Delta x, \qquad \text{where} \qquad C(m, f) = [m(b_{\max})^{\frac{m-1}{m}}]^{-1}. \tag{1.3.9}$$

Then, for every i, k, j we have

$$0 \leq (W_j)_i^k \leq b_{\max}. \tag{1.3.10}$$

Remark 1.3.3. It is interesting to note that $C(1, f) = 1$, which means that we recover the expected restriction $\Delta t \leq \Delta x$ for the linear case. In the literature, this kind of condition use to be called **CFL** condition.

Proof. On each time step we have a discrete harmonic extension problem and so, it is sufficient to prove the maximum principle in the boundary nodes and therefore the interior nodes are automatically smaller than them. The proof is done by induction on each time step. It is trivial that

$$0 \le (W_0)_i^k \le b_{\max}. \qquad \text{for all} \qquad i = 0, \dots, I \qquad \text{and} \qquad k = 0, \dots, K.$$

Now we assume that

$$0 \le (W_{j-1})_i^k \le b_{\max} \qquad \text{for all} \qquad i = 0, \dots, I \qquad \text{and} \qquad k = 0, \dots, K. \tag{1.3.11}$$

We recall that for all $i = 1, \dots, (I-1)$ we have

$$[(W_j)_i^0]^{1/m} = \frac{\Delta t}{\Delta x}\left((W_{j-1})_i^1 - (W_{j-1})_i^0\right) + [(W_{j-1})_i^0]^{1/m}.$$

Changing variables to $(U_j)_i^k = [(W_j)_i^k]^{1/m}$ in the equation above we obtain that

$$(U_j)_i^0 = \frac{\Delta t}{\Delta x}\left([(U_{j-1})_i^1]^m - [(U_{j-1})_i^0]^m\right) + (U_{j-1})_i^0. \tag{1.3.12}$$

We use the Mean Value Theorem for the function $\varphi(s) = s^m$ to get that $[(U_{j-1})_i^1]^m - [(U_{j-1})_i^0]^m = \left((U_{j-1})_i^1 - (U_{j-1})_i^0\right)\varphi'(\xi)$, for some $\xi \in [(U_{j-1})_i^1, (U_{j-1})_i^0]$. This fact allows us to rewrite (1.3.12) as

$$(U_j)_i^0 = m\xi^{m-1}\frac{\Delta t}{\Delta x}(U_{j-1})_i^1 + \left[1 - m\xi^{m-1}\frac{\Delta t}{\Delta x}\right](U_{j-1})_i^0. \tag{1.3.13}$$

By the induction hypothesis (1.3.11) we have that

$$\varphi'(\xi) = m\xi^{m-1} \le m(\max_{i,k}\{(U_{j-1})_i^k\})^{m-1} \le m(b_{\max})^{\frac{m-1}{m}}.$$

Therefore, using (1.3.9) we get observe that $\varphi'(\xi)\frac{\Delta t}{\Delta x} \le 1$. Hence

$$\begin{aligned}
|(U_j)_i^0| &\le m\xi^{m-1}\frac{\Delta t}{\Delta x}|(U_{j-1})_i^1| + \left[1 - m\xi^{m-1}\frac{\Delta t}{\Delta x}\right]|(U_{j-1})_i^0| \\
&\le m\xi^{m-1}\frac{\Delta t}{\Delta x}(b_{\max})^{1/m} + \left[1 - m\xi^{m-1}\frac{\Delta t}{\Delta x}\right](b_{\max})^{1/m} \\
&= (b_{\max})^{1/m},
\end{aligned}$$

that is, $|(W_j)_i^0| \le b_{\max}$. The same argument holds to prove that $(U_j)_i^0 \ge 0$ provided a nonnegative f. □

Corollary 1.3.4 (Existence and uniqueness). *If $\Delta t \le C(m, f)\Delta x$ for the constant $C(m, f)$ defined in (1.3.9), then discrete problem (1.3.3) has a unique solution.*

Proof. Since we are working with a linear system of equations with the same number of variables and equations, at each time step $j = 0, \dots, J$, existence and uniqueness are equivalent. We are going to prove uniqueness

Let $(W_j)_i^k$ and $(V_j)_i^k$ be two solutions of (1.3.3) with the same initial condition. We define $(Y_j)_i^k = (W_j)_i^k - (V_j)_i^k$ and observe that $(Y_0)_i^k$ satisfies (1.3.3) when $j = 0$ with $f \equiv 0$ for all $i = 0, \dots, I$ and $k = 0, \dots, K$. Then the discrete maximum principle yields $0 \leq (Y_0)_i^k \leq 0$, that is, $(Y_0)_i^k = 0$ for all $i = 0, \dots, I$ and $k = 0, \dots, K$. Proceeding by induction on j we get that $(Y_j)_i^k = 0$, and therefore, $(W_j)_i^k = (V_j)_i^k$ for all $i = 0, \dots, I$, $k = 0, \dots, K$ and $j = 0, \dots, J$. $\square$

1.3.5 Convergence of the numerical solution

Our goal is to prove that the solutions of the numerical method (1.3.3) converge to the solutions of extension problem (1.2.3) posed in the bounded domain Ω. Since we are originally interested in the solution at the boundary Γ_d, we have two options in order to define the define de error of the numerical method. Let us use the notation $(U_j)_i^k := [(W_j)_i^k]^{1/m}$. The values $(U_j)_i^0$ are the approximation of the solution $u(x_i, t_j)$ of (1.2.1) when $x_i \in [-X, X]$ and $t_j \in [0, T]$. In this way, we define the following concepts of errors for every time step $j = 0, \dots, J$:

Error in the variable of the extension problem (1.2.1)**:**

$$(f_j)_i^k = w(x_i, y_k, t_j) - (W_j)_i^k, \quad F_j = \max_{i,k} |(f_j)_i^k|. \tag{1.3.14}$$

Error in the variable of the original problem (1.2.2)**:**

$$(e_j)_i^k = (w(x_i, y_k, t_j))^{1/m} - (U_j)_i^k, \quad E_j = \max_{i,k} |(e_j)_i^k|. \tag{1.3.15}$$

In fact, it is enough to have control of the error defined in (1.3.15), and this will give us also control of the error defined in (1.3.14). Let us use the notation $(w_j)_i^j := w(x_i, y_k, t_j)$ and $(u_j)_i^j := (w(x_i, y_k, t_j))^{1/m}$. Then, by the Mean Value theorem, we get

$$\begin{aligned} (f_j)_i^k &= (w_j)_i^k - (W_j)_i^k = [(u_j)_i^j]^m - [(U_j)_i^k]^m \\ &= m\xi^{m-1}\big[(u_j)_i^j - (U_j)_i^k\big] \\ &= m\xi^{m-1}(e_j)_i^k, \end{aligned} \tag{1.3.16}$$

for some $\xi \in [(u_j)_i^j, (U_j)_i^k]$. Since both $(u_j)_i^j$ and $(U_j)_i^k$ are less than or equal to $b_{\max}^{1/m}$ we have that $|(f_j)_i^k| \leq m b_{\max}^{\frac{m-1}{m}} |(e_j)_i^k|$ and therefore $F_j \leq m b_{\max}^{\frac{m-1}{m}} E_j$.

Theorem 1.3.5 (Convergence of the numerical method)**.** *Let $m \geq 1$. Let also w be the solution to Problem* (1.2.3) *and $(W_j)_i^k$ be the solution of the system* (1.3.3)-(1.3.4)*. Assume that*

$$\Delta t \leq C(m,f)\Delta x \qquad \textit{where} \qquad C(m,f) = [m(b_{\max})^{\frac{m-1}{m}}]^{-1} \tag{1.3.17}$$

Then numerical method is convergent. Moreover, we have the following rate of convergences in terms of $\Delta x, \Delta t$ and E_j defined in (1.3.15)*:*

$$E_j = O(\Delta x + \Delta t) \qquad \textit{for} \qquad j = 1, \ldots, J.$$

Proof. As in the local truncation error, the election of homogeneous Dirichlet boundary conditions in Γ_h in the numerical method give us error zero there.

Lets us denote by $(F_B)_j$ and $(E_B)_j$ to the maximum error in the boundary nodes at each time step $j = 0, \ldots, J$, that is,

$$(F_B)_j := \max_{0 \leq i \leq I} |(f_j)_i^0|, \qquad (E_B)_j := \max_{0 \leq i \leq I} |(e_j)_i^0|.$$

We have chosen a second order approximation for the laplacian, and thus, it is well known that the error is propagated to the interior of the domain as Δx^2. Therefore, for all $j = 0, \ldots, J$, we have that

$$F_j = \max\{(F_B)_j, O(\Delta x^2)\} \leq (F_B)_j + O(\Delta x^2) \tag{1.3.18}$$

In this way, we only need to control the error at the boundary Γ_d at each time step in order to control the error in the whole domain Ω.

For $0 \leq i \leq I$ we have the following equations coming from the local truncation error and the numerical method:

$$\begin{aligned}
[(w_j)_i^0]^{1/m} &= \frac{\Delta t}{\Delta x}\left[(w_{j-1})_i^1 - (w_{j-1})_i^0\right] + [(w_{j-1})_i^0]^{1/m} - (\tau_{j-1})_i^0, \\
[(W_j)_i^0]^{1/m} &= \frac{\Delta t}{\Delta x}\left[(W_{j-1})_i^1 - (W_{j-1})_i^0\right] + [(W_{j-1})_i^0]^{1/m}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
(u_j)_i^0 &= \frac{\Delta t}{\Delta x}\left[(w_{j-1})_i^1 - (w_{j-1})_i^0\right] + (u_{j-1})_i^0 - (\tau_{j-1})_i^0, \\
(U_j)_i^0 &= \frac{\Delta t}{\Delta x}\left[(W_{j-1})_i^1 - (W_{j-1})_i^0\right] + (U_{j-1})_i^0.
\end{aligned}$$

Subtracting them and recalling that $(e_j)_i^k := (u_j)_i^k - (U_j)_i^k$ and $(f_j)_i^k := (w_j)_i^k - (W_j)_i^k$, we get

$$\begin{aligned}(e_j)_i^0 &= \frac{\Delta t}{\Delta x}\Big[\big((w_{j-1})_i^1 - (W_{j-1})_i^1\big) - \big((w_{j-1})_i^0 - (W_{j-1})_i^0\big)\Big] + (e_{j-1})_i^0 - (\tau_{j-1})_i^0 \\ &= \frac{\Delta t}{\Delta x}\Big[(f_{j-1})_i^1 - (f_{j-1})_i^0\Big] + (e_{j-1})_i^0 - (\tau_{j-1})_i^0\end{aligned}.$$

If $(e_{j-1})_i^0 = (f_{j-1})_i^0 = 0$, then (1.3.16), (1.3.18) and (1.3.17) yields

$$\begin{aligned}|(e_j)_i^0| &\le \frac{\Delta t}{\Delta x}|(f_{j-1})_i^1| + |(\tau_{j-1})_i^0| \le \frac{\Delta t}{\Delta x}\left((F_B)_{j-1} + O(\Delta x^2)\right) + |(\tau_{j-1})_i^0| \\ &\le \frac{\Delta t}{\Delta x} m(b_{\max})^{\frac{m-1}{m}}(E_B)_{j-1} + |(\tau_{j-1})_i^0| + O(\Delta t\Delta x) \\ &\le E_{j-1} + \max_{i,j}\{|(\tau_{j-1})_i^0|\} + O(\Delta t\Delta x)\end{aligned}$$

If $(e_{j-1})_i^0 \neq 0$, identity (1.3.16) and the value of the constant (1.3.9) together with hypothesis (1.3.17) show that

$$\frac{(e_{j-1})_i^k}{(f_{j-1})_i^k} \ge \left[m(b_{\max})^{\frac{m-1}{m}}\right]^{-1} \ge \frac{\Delta t}{\Delta x}.$$

Then, using (1.3.18), we obtain that

$$\begin{aligned}|(e_j)_i^0| &= \frac{\Delta t}{\Delta x}(f_{j-1})_i^1 + \left[\frac{(e_{j-1})_i^0}{(f_{j-1})_i^0} - \frac{\Delta t}{\Delta x}\right](f_{j-1})_i^0 - (\tau_{j-1})_i^0 \\ &\le \frac{\Delta t}{\Delta x}|(f_{j-1})_i^1| + \left[\frac{(e_{j-1})_i^0}{(f_{j-1})_i^0} - \frac{\Delta t}{\Delta x}\right]|(f_{j-1})_i^0| + |(\tau_{j-1})_i^0| \\ &\le \frac{\Delta t}{\Delta x}\Big(|(f_{j-1})_i^1| - |(f_{j-1})_i^0|\Big) + |(e_{j-1})_i^0| + |(\tau_{j-1})_i^0| \\ &\le K\Delta t\Delta x + E_{j-1} + \max_{i,j}\{|(\tau_{j-1})_i^0|\},\end{aligned}$$

for some constant $K > 0$. Hence, since the right hand side of the above estimate is independent on i we have that

$$E_j \le K\Delta t\Delta x + E_{j-1} + \max_{i,j}\{|(\tau_{j-1})_i^0|\} \qquad \text{for all} \qquad j = 1,\dots,J.$$

Then,

$$\begin{aligned}E_J &\le K\Delta t\Delta x + E_{J-1} + \max_{i,j}\{|(\tau_{j-1})_i^0|\} \\ &\le KJ\Delta t\Delta x + E_0 + J\max_{i,j}\{|(\tau_{j-1})_i^0|\}.\end{aligned}$$

We recall that $E_0 = \max_i |w(x_i,0,0) - (W_0)_i^0| = \max_i |f^m(x_i) - f^m(x_i)| = 0$. Moreover, by Theorem 1.3.1 we have that $\max_{i,j}\{|(\tau_{j-1})_i^0|\} = O(\Delta t(\Delta x + \Delta t))$. Therefore, since

$J\Delta t = T$ by the time discretization definition, we conclude that $E_J \leq O(\Delta x + \Delta t)$. $\square$

1.3.6 Properties of the scheme

We present now some properties that can be deduced from the numerical scheme (1.3.3)-(1.3.4). They are the analogues of some of the energy estimates presented in [37, 38] for the fractional porous medium equation. The first one is a consequence of the discrete maximum principle.

Corollary 1.3.6 (Comparison principle)**.** *Let $m \geq 1$. Consider f and g nonnegative such that $f(x) \geq g(x)$ $\forall x \in [-X, X]$. Let also $(W_j)_i^k := [(U_j)_i^k]^m$ and $(Z_j)_i^k := [(V_j)_i^k]^m$ be the solutions of Problem (1.3.3)-(1.3.4) with initial data f and g respectively. If $\Delta t \leq C(m, f)\Delta x$ then,*

$$(W_j)_i^k \geq (Z_j)_i^k \quad \forall i, j, k.$$

Proof. Let $(H_j)_i^k$ be the solution of Problem (1.3.3)-(1.3.4) with a nonnegative initial data given by $h = (f^m - g^m)^{1/m}$. By the Discrete Maximum Principle 1.3.2, we have that $(H_j)_i^k \geq 0$. At time $j = 0$ the scheme is linear (see (1.3.4)) and therefore $(W_0)_i^k - (Z_0)_i^k = (H_0)_i^k \geq 0$. Proceeding by induction assume that $(W_j)_i^k - (Z_j)_i^k \geq 0$, or equivalently $(U_j)_i^k - (V_j)_i^k \geq 0$. The following two identities hold:

$$\begin{aligned}
(U_{j+1})_i^0 &= \frac{\Delta t}{\Delta x}\left([(U_j)_i^1]^m - [(U_j)_i^0]^m\right) + (U_j)_i^0, \\
(V_{j+1})_i^0 &= \frac{\Delta t}{\Delta x}\left([(V_j)_i^1]^m - [(V_j)_i^0]^m\right) + (V_j)_i^0.
\end{aligned}$$

Then, for some $\xi_0 \in [(U_j)_i^0, (V_j)_i^0]$ and $\xi_1 \in [(U_j)_i^1, (V_j)_i^1]$, the mean value theorem yields

$$\begin{aligned}
&(U_{j+1})_i^0 - (V_{j+1})_i^0 \\
&= \frac{\Delta t}{\Delta x}\left(\left([(U_j)_i^1]^m - [(V_j)_i^1]^m\right) - \left([(U_j)_i^0]^m - [(V_j)_i^0]^m\right)\right) + (U_j)_i^0 - (V_j)_i^0 \\
&= \frac{\Delta t}{\Delta x}\left(m\xi_1^{m-1}\left((U_j)_i^1 - (V_j)_i^1\right) - m\xi_0^{m-1}\left((U_j)_i^0 - (V_j)_i^0\right)\right) + (U_j)_i^0 - (V_j)_i^0 \\
&= \frac{\Delta t}{\Delta x} m\xi_1^{m-1}\left((U_j)_i^1 - (V_j)_i^1\right) + \left[1 - \frac{\Delta t}{\Delta x} m\xi_0^{m-1}\right]\left((U_j)_i^0 - (V_j)_i^0\right).
\end{aligned}$$

By the induction hypothesis we have that $(U_j)_i^k - (V_j)_i^k \geq 0$. Moreover, the CFL condition (1.3.9) ensures that $\frac{\Delta t}{\Delta x} m\xi_0^{m-1} \leq \frac{\Delta t}{\Delta x} m(b_{\max})^{\frac{m-1}{m}} \leq 1$. Therefore, all the terms in the last identity are positive and then $(U_{j+1})_i^0 - (V_{j+1})_i^0 \geq 0$. The Discrete Maximum Principle 1.3.2 concludes the proof. $\square$

The following L^1-contraction property is the discrete version of the one presented in [37] (Theorem 6.2).

Theorem 1.3.7 (Discrete L^1-contraction)**.** *Under the assumptions of Corollary 1.3.6 let* $(U_j)_i^k = [(W_j)_i^k]^{1/m}$ *and* $(V_j)_i^k = [(Z_j)_i^k]^{1/m}$*. Then,*

$$\sum_{i=1}^{I-1} \left[(U_j)_i^0 - (V_j)_i^0\right] \le \sum_{i=1}^{I-1} \left[(U_{j-1})_i^0 - (V_{j-1})_i^0\right], \tag{1.3.19}$$

for all $j = 1, \dots, J$.

Proof. We star by proving (1.3.19) for $j = 1$ and proceed by induction to prove it for any $j = 2, \dots, J$. The following identities hold for every $i = 1, \dots, I-1$,

$$\begin{aligned}
(U_1)_i^0 &= \frac{\Delta t}{\Delta x}\big((W_0)_i^1 - f^m(x_i)\big) + f(x_i),\\
(V_1)_i^0 &= \frac{\Delta t}{\Delta x}\big((Z_0)_i^1 - g^m(x_i)\big) + g(x_i).
\end{aligned}$$

Subtracting them and summing for all $i = 1, \dots, I-1$, we get

$$\begin{aligned}
\sum_{i=1}^{I-1}[(U_1)_i^1 - (V_1)_i^1] =& \sum_{i=1}^{I-1}[f(x_i) - g(x_i)]\\
&+ \frac{\Delta t}{\Delta x}\left(\sum_{i=1}^{I-1}[(W_0)_i^1 - (Z_0)_i^1] - \sum_{i=1}^{I-1}[f^m(x_i) - g^m(x_i)]\right).
\end{aligned} \tag{1.3.20}$$

We need to show that $\sum_{i=1}^{I-1}[(W_1)_i^1 - (Z_1)_i^1] \le \sum_{i=1}^{I-1}[f^m(x_i) - g^m(x_i)]$. From (1.3.3) we have that for any $k = 1, \dots, K-1$ we have that

$$\begin{aligned}
4(W_j)_i^k &= (W_j)_{i+1}^k + (W_j)_{i-1}^k + (W_j)_i^{k+1} + (W_j)_i^{k-1}\\
4(Z_j)_i^k &= (Z_j)_{i+1}^k + (Z_j)_{i-1}^k + (Z_j)_i^{k+1} + (Z_j)_i^{k-1}
\end{aligned}$$

Define $(H_j)_i^k := (W_j)_i^k - (Z_j)_i^k$. By comparison principle given in Corollary 1.3.6 we have that $(H_j)_i^k \ge 0$. Subtracting the equations above and summing on i we get the following identity valid for all $k = 1, \dots, K-1$:

$$\begin{aligned}
\sum_{i=1}^{I-1}(H_j)_i^{k-1} &= 4\sum_{i=1}^{I-1}(H_j)_i^k - \sum_{i=1}^{I-1}(H_j)_{i+1}^k - \sum_{i=1}^{I-1}(H_j)_{i-1}^k - \sum_{i=1}^{I-1}(W_j)_i^{k+1}\\
&= 4\sum_{i=1}^{I-1}(H_j)_i^k - \sum_{i=2}^{I-1}(H_j)_i^k - \sum_{i=1}^{I-2}(H_j)_i^k - \sum_{i=1}^{I-1}(H_j)_i^{k+1}\\
&= (H_j)_1^k + (H_j)_{I-1}^k + 2\sum_{i=1}^{I-1}(H_j)_i^k - \sum_{i=1}^{I-1}(H_j)_i^{k+1}.
\end{aligned}$$

where the second identity is obtained from the boundary conditions $(W_j)_I^k = (W_j)_0^k = 0$ for all $k = 0, \dots, K$ of (1.3.3). Now consider the relation above for $k = K-1$ and recall that $(W_j)_i^K = (Z_j)_i^K = (H_j)_i^K = 0$ for all $i = 0, \dots, I$ to get

$$\begin{aligned}\sum_{i=1}^{I-1}(H_j)_i^{K-2} &= (H_j)_1^{K-1} + (H_j)_{I-1}^{K-1} + 2\sum_{i=1}^{I-1}(H_j)_i^{K-1} - \sum_{i=1}^{I-1}(H_j)_i^K \\ &\geq \sum_{i=1}^{I-1}(H_j)_i^{K-1}.\end{aligned}$$

Proceeding by induction, assume that $\sum_{i=1}^{I-1}(H_j)_i^{k-1} \geq \sum_{i=1}^{I-1}(H_j)_i^k$. Then

$$\begin{aligned}\sum_{i=1}^{I-1}(H_j)_i^{k-2} &= (H_j)_1^{k-1} + (H_j)_{I-1}^{k-1} + 2\sum_{i=1}^{I-1}(H_j)_i^{k-1} - \sum_{i=1}^{I-1}(H_j)_i^k \\ &\geq 2\sum_{i=1}^{I-1}(H_j)_i^{k-1} - \sum_{i=1}^{I-1}(H_j)_i^{k-1} = \sum_{i=1}^{I-1}(H_j)_i^{k-1},\end{aligned}$$

that is,

$$\sum_{i=1}^{I-1}[(W_j)_i^{k-1} - (Z_j)_i^{k-1}] \geq \sum_{i=1}^{I-1}[(W_j)_i^k - (Z_j)_i^k] \qquad \text{for all} \qquad k = 1, \dots, K,$$

which in particular shows that $\sum_{i=1}^{I-1}[f^m(x_i) - g^m(x_i)] \geq \sum_{i=1}^{I-1}[(W_j)_i^1 - (Z_j)_i^1]$. □

Remark 1.3.8. We get the mass decay property

$$\sum_{i=i}^{I-1}(U_j)_i^0 \leq \sum_{i=i}^{I-1}(U_{j-1})_i^0$$

as consequence of Theorem 1.3.7 by choosing the solution $(Z_j)_i^k$ with initial condition $g \equiv 0$. The lack of conservation of mass is due to the artificial homogeneous Dirichlet boundary condition imposed. In fact, the following result shows that when this condition is not imposed, the expected conservation of mass holds.

Theorem 1.3.9 (Conservation of mass)**.** *Let $m \geq 1$. Consider the solution $(W_j)_i^k$ to (1.3.3)-(1.3.4) posed in $\Omega = \mathbb{R} \times \{y \geq 0\}$ with initial condition f. If*

$$\sum_{i\in\mathbb{Z}} f(x_i) < \infty, \tag{1.3.21}$$

then for all $j = 1, \dots, J$ we have that

$$\sum_{i\in\mathbb{Z}}(U_j)_i^0 = \sum_{i\in\mathbb{Z}}(U_{j-1})_i^0. \tag{1.3.22}$$

Proof. The following identity holds for every $i \in \mathbb{Z}$:

$$(U_1)_i^0 = \frac{\Delta t}{\Delta x}\left[(W_0)_i^1 - (W_0)_i^0\right] + (U_0)_i^0.$$

Summing up we get

$$\sum_{i\in\mathbb{Z}}(U_1)_i^0 = \frac{\Delta t}{\Delta x}\left[\sum_{i\in\mathbb{Z}}(W_0)_i^1 - \sum_{i\in\mathbb{Z}}(W_0)_i^0\right] + \sum_{i\in\mathbb{Z}}(U_0)_i^0. \tag{1.3.23}$$

We need to prove that $\sum_{i\in\mathbb{Z}}(W_0)_i^1 = \sum_{i\in\mathbb{Z}}(W_0)_i^0$ to conclude the proof. From the numerical scheme (1.3.3) we know that for every $k \in \mathbb{Z}_{\geq 0}$ we have that

$$(W_0)_i^k + (W_0)_{i-1}^k + (W_0)_i^{k+1} + (W_0)_i^{k-1} = 4(W_0)_i^k,$$

which yields

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(W_0)_i^{k-1} &= 4\sum_{i\in\mathbb{Z}}(W_0)_i^k - \sum_{i\in\mathbb{Z}}(W_0)_{i+1}^k - \sum_{i\in\mathbb{Z}}(W_0)_{i-1}^k - \sum_{i\in\mathbb{Z}}(W_0)_i^{k+1}\\ &= 2\sum_{i\in\mathbb{Z}}(W_0)_i^k - \sum_{i\in\mathbb{Z}}(W_0)_i^{k+1}.\end{aligned} \tag{1.3.24}$$

Now, let us write $a_k := \sum_{i\in\mathbb{Z}}(W_0)_i^k$ and note that by (1.3.21) we have

$$a_0 = \sum_{i\in\mathbb{Z}}(W_0)_i^0 = \sum_{i\in\mathbb{Z}} f^m(x_i) \leq (b_{\max})^{\frac{m-1}{m}}\sum_{i\in\mathbb{Z}} f(x_i) < \infty.$$

Then, identity (1.3.24) can be interpreted as the following sequence defined in a recurrence form,

$$a_{k+2} = 2a_{k+1} - a_k \qquad \text{with} \qquad a_0 = \sum_{i\in\mathbb{Z}} f^m(x_i).$$

The general solution of this recurrence is $a_k = a_0 + k(a_1 - a_0)$, but the formula requires an initial value of a_1. If $a_1 > a_0$ then identity (1.3.23) will implies that $\sum_{i\in\mathbb{Z}}(U_1)_i^0 > \sum_{i\in\mathbb{Z}}(U_0)_i^0$ which is incompatible with Remark 1.3.8. On the other hand, if $a_1 < a_0$, we can choose $k > \frac{a_0}{a_0-a_1}$ to get that $\sum_{i\in\mathbb{Z}}(W_0)_i^k = a_k < a_0 + \frac{a_0}{a_0-a_1}(a_1 - a_0) < 0$. Then there exists $i \in \mathbb{Z}$ such that $(W_0)_i^k < 0$ which is a contradiction with the discrete maximum principle given in Theorem 1.3.2. So the only possibility is

$$\sum_{i=-\infty}^{\infty}(W_0)_i^1 = a_1 = a_0 = \sum_{i=-\infty}^{\infty}(W_0)_i^0,$$

which concludes the proof. $\square$

1.4 Convergence towards the solution in the whole space.

Regularity of the solution of (1.2.3) which is posed in a bounded domain Ω is a crucial step in the proof of convergence of Theorem 1.3.5. This kind of regularity issues has been recently studied by Bonforte and Vázquez in [18]. By the time when the results of this chapter where found, regularity results for solutions of (1.2.3) in bounded domains were unknown. The only knowledge about regularity was for solutions of (1.2.2) posed in $\mathbb{R}^{N+1}_+$ (see [82]).

In this section we give a rate of convergence in terms of the size of Ω for the numerical solutions given by (1.3.3)-(1.3.4) posed in the bounded domain Ω toward the solution of (1.2.2) posed in $\mathbb{R}^{N+1}_+$. Since we are going to work with solutions in $\mathbb{R}^{N+1}_+$ there are no regularity difficulties to deal with.

We will compare the solution to the numerical scheme (1.3.3)-(1.3.4) posed in the bounded domain $[-X, X]\times[0, X]$ with the theoretical solution to (1.2.2) posed in $\mathbb{R}^{N+1}_+$. Obviously, the comparison is only done where the numerical scheme is well defined.

A new difficulty appears with this comparison. Now $w \neq 0$ in Γ_h and so we can not expect to have convergence for a fixed domain. The idea is making $\Omega \longrightarrow \mathbb{R}^{N+1}_+$ as $\Delta x \longrightarrow 0$ with a certain rage of convergence. Note that an extra error coming from the lateral boundary Γ_h will be introduced.

In [79], an upper bound for the solution w of (1.2.2) posed in $\mathbb{R}^{N+1}_+$ with compactly supported bounded initial data is found. The tool is the so called Barenblatt solutions of problem (1.2.1). The upped bound is,

$$u_B(x,t) = (t+1)^{-\alpha}\phi(|x|(t+1)^{-\beta}),$$

with $\alpha = \frac{N}{N(m-1)+1}$ and $\beta = \frac{1}{N(m-1)+1}$ and a function ϕ such that $\phi(\xi) \leq C|\xi|^{-N-1}$ for some constant $C > 0$. Since $-\alpha + \beta(N+1) = \beta$, we have the following upper bound for w in Γ_h depending only on X and T,

$$\begin{aligned} u_B(X,t) &= (t+1)^{-\alpha}\phi(|X|(t+1)^{-\beta}) \\ &\leq C(t+1)^{-\alpha+\beta(N+1)}\frac{1}{X^{N+1}} \\ &\leq C(t+1)^{\beta}\frac{1}{X^{N+1}} \\ &\leq C(T+1)^{\beta}\frac{1}{X^{2}} \end{aligned}$$

In other words, the solution in the whole space introduce an extra error since it is different than zero in Γ_h. We would like to force this error to be comparable or smaller than the error introduced by the discretization. We impose the following condition in the domain,

$$|X| \geq K\frac{1}{\Delta t}, \tag{1.4.1}$$

for some constant $K > 0$. In this way we can adapt the proofs of Theorem 1.3.1 and Theorem 1.3.5 to obtain convergence.

In Theorem 1.3.1, the local truncation error in the interior of Ω nodes and in Γ_d still being the same but is not zero anymore in Γ_h. Now if $(x_i, y_k) \in \Gamma_h$, (1.4.1) shows that

$$|(\tau_j)_i^k| = |(w_j)_i^k| \leq C\frac{1}{|X|^2} \leq O(\Delta t^2).$$

In Theorem 1.3.5, again the only change is that the error at Γ_h is not zero anymore. But in view of (1.4.1), if $(x_i, y_k) \in \Gamma_h$ we have that

$$|(e_j)_i^k| = |(w_j)_i^k - (W_j)_i^k| = |(w_j)_i^k| \leq O(\Delta t^2).$$

And so $E_J = O(\Delta x + \Delta t)$.

We get then the next result,

Theorem 1.4.1. *Let w be the solution to Problem (1.2.2) and $(W_j)_i^k$ be the solution to system (1.3.3)-(1.3.4) posed in the bounded domain $\Omega = [-X, X] \times [0, X]$ with $m \geq 1$ and smooth and compactly supported initial data f. Assume that:*

(a) For the constant $C(m, f)$ defined in (1.3.9) we have that

$$\Delta t \leq C(m, f)\Delta x.$$

(b) There exists a constant $K > 0$ such that

$$|X| \geq K\frac{1}{\Delta t}.$$

Then

$$\max_{i,j,k} |w(x_i, y_k, t_j) - (W_j)_i^k| = O(\Delta x + \Delta t).$$

Remark 1.4.2. Condition (b) of Theorem 1.4.1 implies that $|X| \overset{\Delta t \to 0}{\longrightarrow} \infty$ and so $\Omega \overset{\Delta t \to 0}{\longrightarrow} \mathbb{R}$.

1.5 More general fractional diffusion equations

The numerical scheme (1.3.3)-(1.3.4) can be easily generalized for a wider class of nonlinearities $\varphi(u)$ instead of u^m, i.e. the so-called generalized fractional porous medium equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^{1/2}\varphi(u(x,t)) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{1.5.1}$$

for $\varphi \in C^2(\mathbb{R})$ such that $\varphi' > 0$ and φ', φ'' locally bounded. We also need existence of $\varphi^{-1} \in C^2(\mathbb{R})$. We can assume that $\varphi(0) = 0$ by adding or subtracting a constant to the function φ. In this case, we have an associated harmonic extension problem

$$\begin{cases} \Delta w(x,y,t) = 0, & x \in \mathbb{R},\ y > 0,\ t > 0, \\ \dfrac{\partial \varphi^{-1}(w)}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y}(x,0,t), & x \in \mathbb{R},\ y = 0,\ t > 0, \\ w(x,0,0) = \varphi(f(x)), & x \in \mathbb{R}. \end{cases} \tag{1.5.2}$$

As before, the equivalence between (1.5.1) and (1.5.2) holds in the sense of trace and harmonic extension, that is $u(x,t) = Tr(\varphi^{-1}(w(x,y,t)))$ and $w(x,y,t) = E(u^m(x,t))$. In [39] they study the theory for this problem with

$$\varphi(u) = \log(u+1),$$

as the natural limit as $m \to 0$ of the problem with

$$\varphi_m(u) = \frac{(u+1)^m - 1}{m}.$$

We can also state the corresponding finite difference scheme associated to (1.5.2) posed in the bounded domain Ω,

$$\begin{cases} \dfrac{(W_j)_{i+1}^k + (W_j)_{i-1}^k + (W_j)_i^{k+1} + (W_j)_i^{k-1} - 4(W_j)_i^k}{\Delta x^2} = 0, & 0 < i < I, 0 < k < K \\ (W_j)_i^0 = \varphi\Big(\dfrac{\Delta t}{\Delta x}\big((W_{j-1})_i^1 - (W_{j-1})_i^0\big) + \varphi^{-1}[(W_{j-1})_i^0]\Big), & \text{if } 0 < i < I \\ (W_j)_i^k = 0, & \text{otherwise,} \end{cases} \tag{1.5.3}$$

where the solution of

$$\begin{cases} \dfrac{(W_0)_{i+1}^k + (W_0)_{i-1}^k + (W_0)_i^{k+1} + (W_0)_i^{k-1} - 4(W_0)_i^k}{\Delta x^2} = 0, & 0 < i < I,\ 0 < k < K \\ (W_0)_i^0 = \varphi(f(x_i)), & \text{if } 0 < i < I, \\ (W_0)_i^k = 0, & \text{otherwise,} \end{cases} \tag{1.5.4}$$

is used to start the numerical method. All the proofs of the results of Section 1.3 can be adapted without any extra effort for the numerical method defined by (1.5.3)-(1.5.4). Of course we need to formulate a generalization related to φ of condition given in (1.3.9). We recall that condition (1.3.9) relays on the positivity of the $1 - \varphi'(\xi)\Delta t/\Delta x$

for $\xi \in [0, \max_x\{f(x)\}]$ that appears in (5.1.22). We define

$$b_{\max} = \max_{x \in \mathbb{R}} \{\varphi(f(x))\}.$$

In this way, assumption (1.3.17) becomes

$$\Delta t \leq C(\varphi, f)\Delta x \qquad \text{where} \qquad C(\varphi, f) = \left(\max_{\nu \in [0, \varphi^{-1}(b_{\max})]} \{\varphi'(\nu)\} \right)^{-1} \tag{1.5.5}$$

1.6 Numerical results

In this section we present two numerical experiments in order to compute the error of the numerical solution. By the time that this research was done, no explicit solution where known for problem (1.2.1). In [53], the author shows explicit solutions for certain choices of m and the power σ of the fractional laplacian $(-\Delta)^{\frac{\sigma}{2}}$. These explicit solutions open a new possibility of computing numerical errors.

1.6.1 First analysis of the error

The first error analysis is done by comparing a sequence of numerical solutions generated by a sequence of decreasing Δx with a numerical solution generated with a very small Δx. The results are presented in Figure 1.3 and Tables 1.2 and 1.1 .

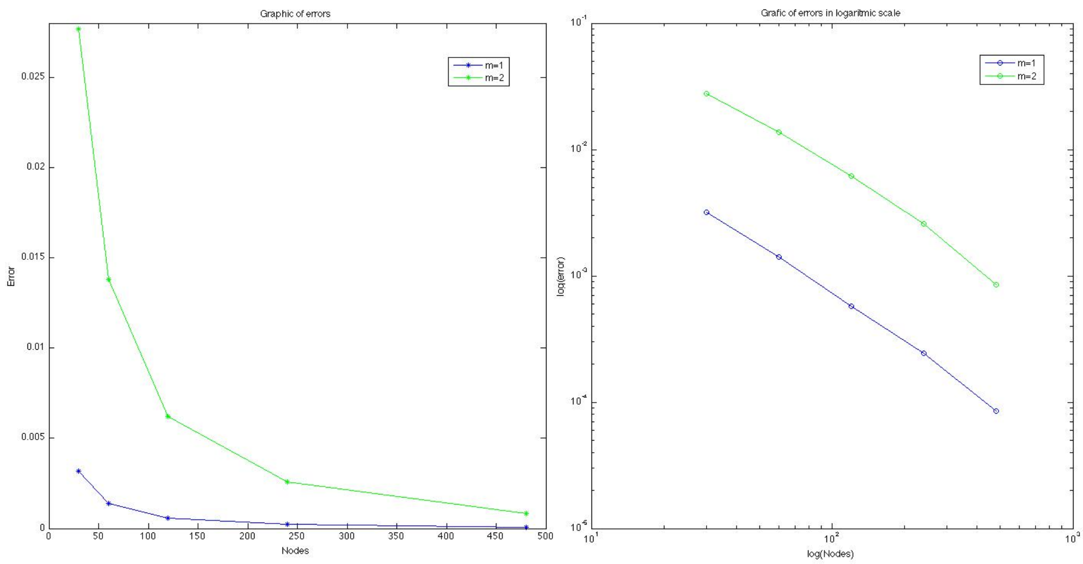

FIGURE 1.3: First analysis of the error for m=1 and m=2

m	$\Delta x = \Delta t$	$Nodes$	$Error$
1	0.2	30	0.0031733
	0.1	60	0.0014071
	0.05	120	0.0005760
	0.025	240	0.0002443
	0.0125	480	0.0000852

TABLE 1.1: First analysis of the error for $m = 1$.

m	$\Delta x = \Delta t$	$Nodes$	$Error$
2	0.2	30	0.0276966
	0.1	60	0.0137975
	0.05	120	0.0061930
	0.025	240	0.0025897
	0.0125	480	0.0008505

TABLE 1.2: First analysis of the error for $m = 2$.

1.6.2 Second analysis of the error

The second way of computing the errors can only be done for the case $m = 1$. The idea is to use the explicit solution problem (1.2.1) posed in $\mathbb{R}^N$, with a Dirac delta as initial data given by

$$v(x,t) = \frac{1}{\pi}\frac{t}{|x|^2 + t^2}.$$

For computational reasons instead of considering the Dirac delta as initial condition, we consider

$$f(x) := v(x,1) = \frac{1}{\pi}\frac{1}{|x|^2 + 1}. \tag{1.6.1}$$

We know that

$$u(x,t) = \frac{1}{\pi}\frac{t+1}{|x|^2 + (t+1)^2} \tag{1.6.2}$$

is the explicit solution of (1.2.1) with initial condition (1.6.1) . We are going to compare the solution given by (1.6.2) for the problem (1.2.1) posed in the whole space with a numerical solution given by (1.3.3)-(1.3.4) computed in a bounded domain. We run numerical experiments for different domains in order to observe the influence of the artificial boundary in the rate of convergence of the error. The errors are obtained by comparing the both the numerical and the real solution in the nodes corresponding to the domain $[-2, 2]$. The consequence of Theorem 1.4.1 is clearly observed in tables 1.3 and 1.4 and figures 1.4 and 1.5. We see that the error in these cases stop decaying properly at some point even the the size of the spatial-time step is smaller and smaller. This is consequence of the error coming from the artificial boundary Γ_h. On the other hand, tables 1.5 and 1.6 and figures 1.6 and 1.7 show that if the domain is big enough, the influence of the error coming from the boundary does not affect to the linear decay of the error with respect to the step. We summarize the results in Figure 1.8 and Figure 1.9.

m	$\Delta x = \Delta t$	$x -$ Nodes	Error
1	2	20	0.0286902
	1	40	0.0039588
	0.5	80	0.0015831
	0.25	160	0.0013261
	0.125	320	0.0012564

TABLE 1.3: Second error analysis for $m = 1$ with $\Omega = [-20, 20] \times [0, 20]$ and $T = 2$.

m	$\Delta x = \Delta t$	$x -$ Nodes	Error
1	2	50	0.0299196
	1	100	0.0050327
	0.5	200	0.0008987
	0.25	400	0.0002929
	0.125	800	0.0002307

TABLE 1.4: Second error analysis for $m = 1$ with $\Omega = [-50, 50] \times [0, 50]$ and $T = 2$.

m	$\Delta x = \Delta t$	$x -$ Nodes	Error
1	2	100	0.0300969
	1	200	0.0051925
	0.5	400	0.0009573
	0.25	800	0.0001934
	0.125	1600	0.0000742

TABLE 1.5: Second error analysis for $m = 1$ with $\Omega = [-100, 100] \times [0, 100]$ and $T = 2$.

m	$\Delta x = \Delta t$	$x -$ Nodes	Error
1	2	200	0.0300995
	1	400	0.0051950
	0.5	800	0.0009598
	0.25	1600	0.0001959
	0.125	3200	0.0000717

TABLE 1.6: Second error analysis for $m = 1$ with $\Omega = [-200, 200] \times [0, 100]$ and $T = 2$.

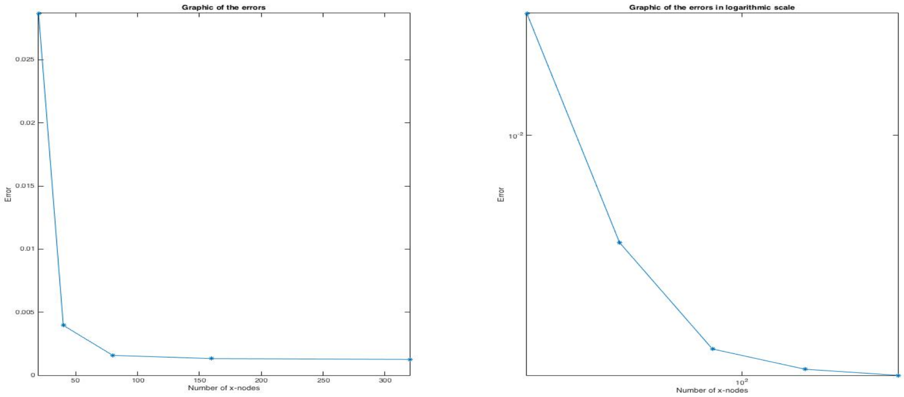

FIGURE 1.4: Second error analysis for $m = 1$ with $\Omega = [-20, 20] \times [0, 20]$ and $T = 2$.

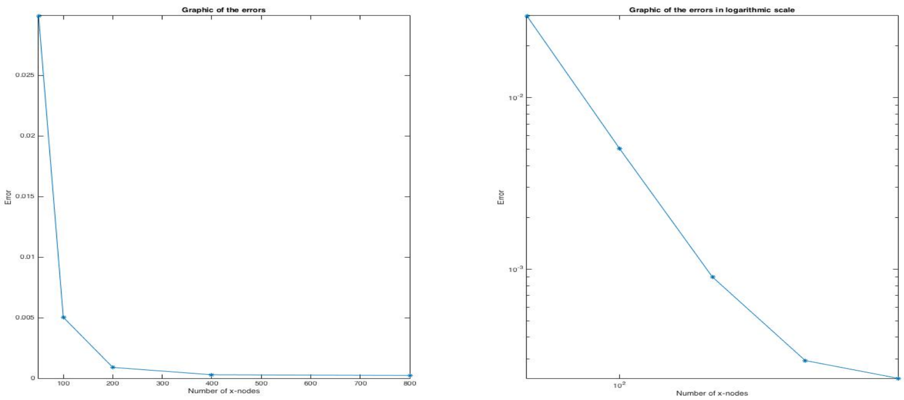

FIGURE 1.5: Second error analysis for $m = 1$ with $\Omega = [-50, 50] \times [0, 50]$ and $T = 2$.

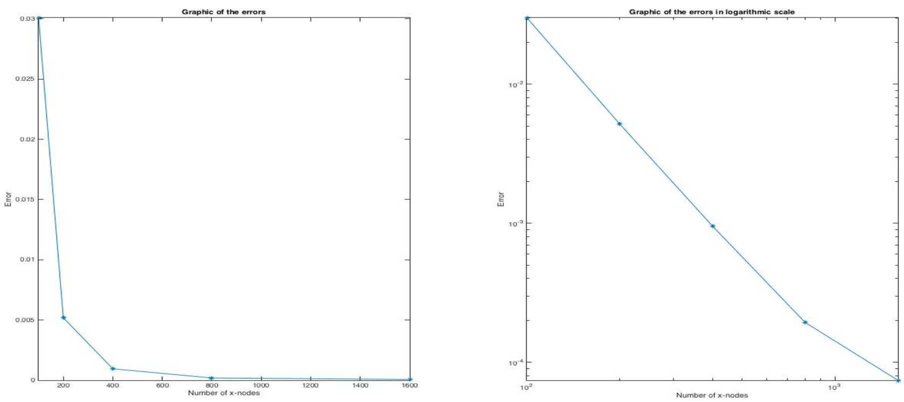

FIGURE 1.6: Second error analysis for $m = 1$ with $\Omega = [-100, 100] \times [0, 100]$ and $T = 2$.

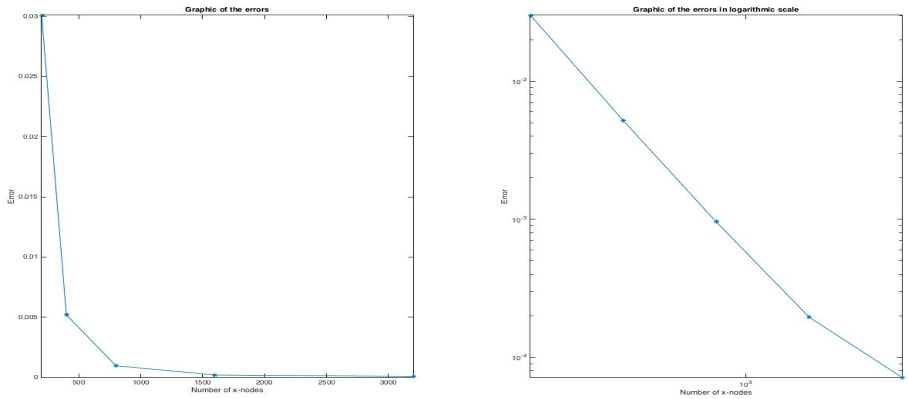

FIGURE 1.7: Second error analysis for $m = 1$ with $\Omega = [-200, 200] \times [0, 100]$ and $T = 2$.

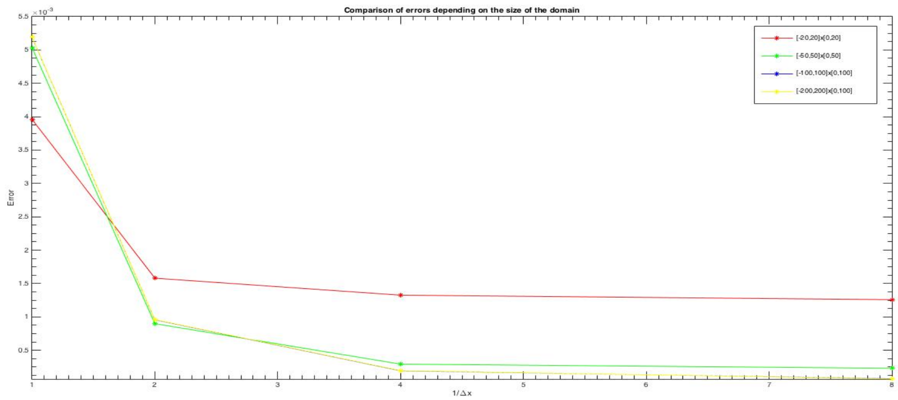

FIGURE 1.8: Comparison of errors at $T = 2$.

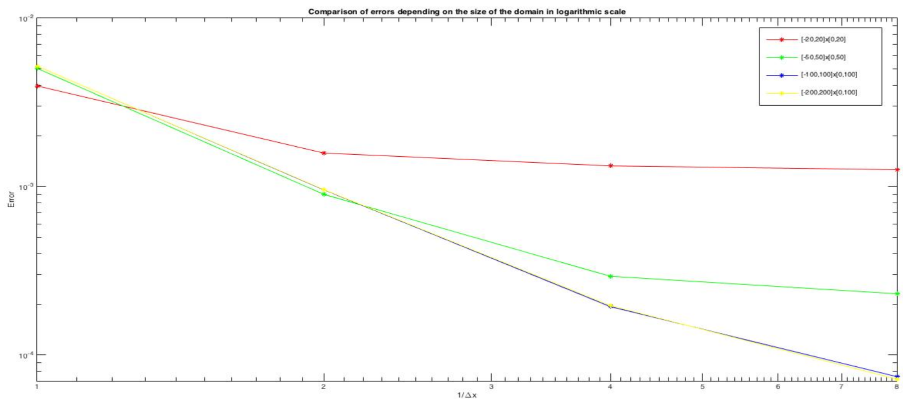

FIGURE 1.9: Comparison of errors at $T = 2$ in logarithmic scale.

1.6.3 Other possible analysis of the error

A third way of computing errors for $m \neq 1$ is possible thanks to the Barenblatt formula introduced by J.L. Vázquez in [79]. In that paper this numerical method is used to

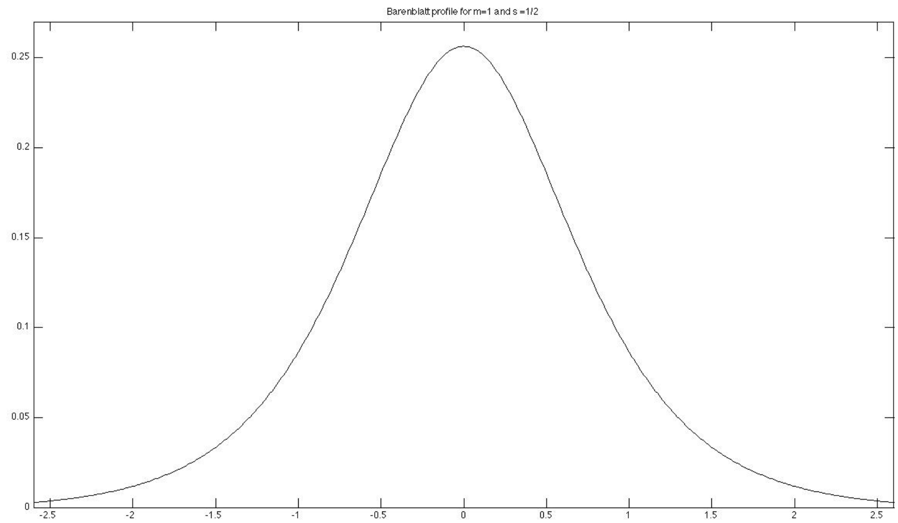

FIGURE 1.10: Computed Barenblatt profiles for $m = 1$ and $s = 1/2$

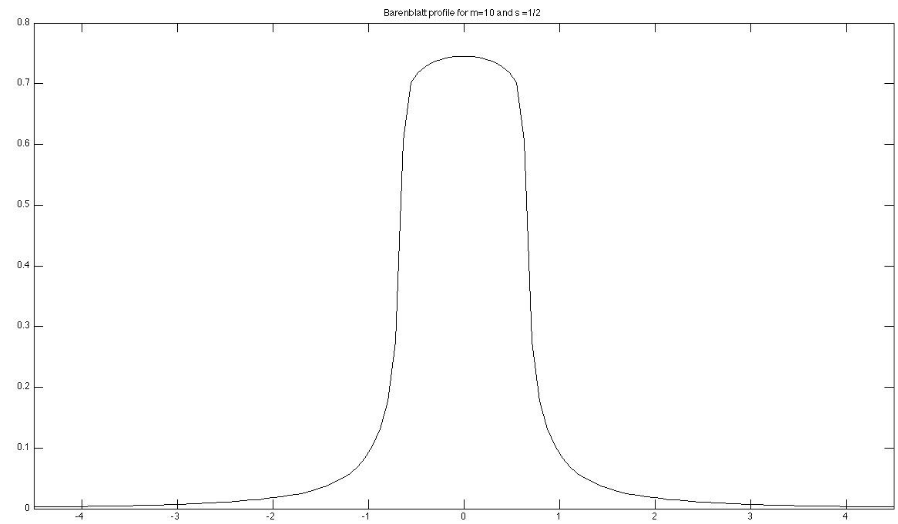

FIGURE 1.11: Computed Barenblatt profiles for $m = 10$ and $s = 1/2$

compute some Barenblatt profiles (See Pictures 1.10 and 1.11)

1.6.4 Graphics of some solutions

We present now some pictures of the numerical solutions obtained with the numerical scheme (1.3.3)-(1.3.4). In all the cases we consider the initial data

$$f(x) = Ce^{-\frac{1}{(1-x)(1+x)}}\chi_{[-1,1]}(x).$$

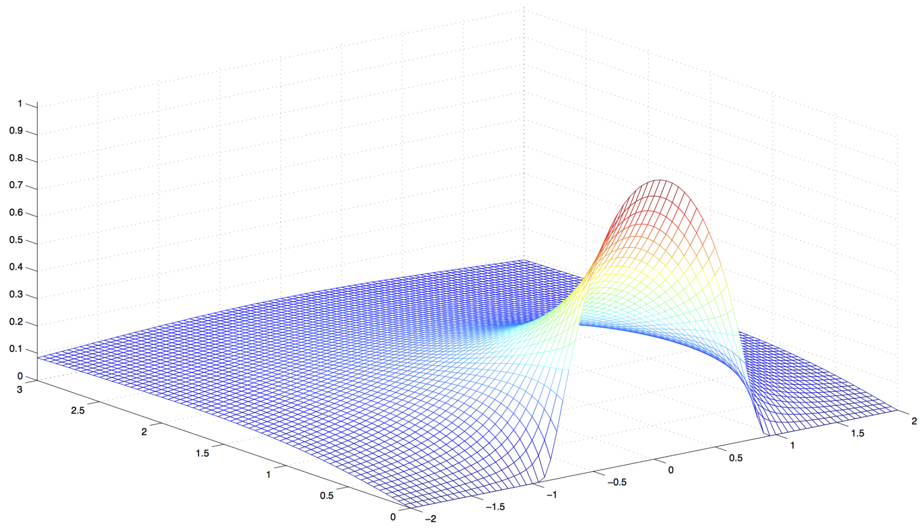

FIGURE 1.12: Numerical solution for $m = 1$

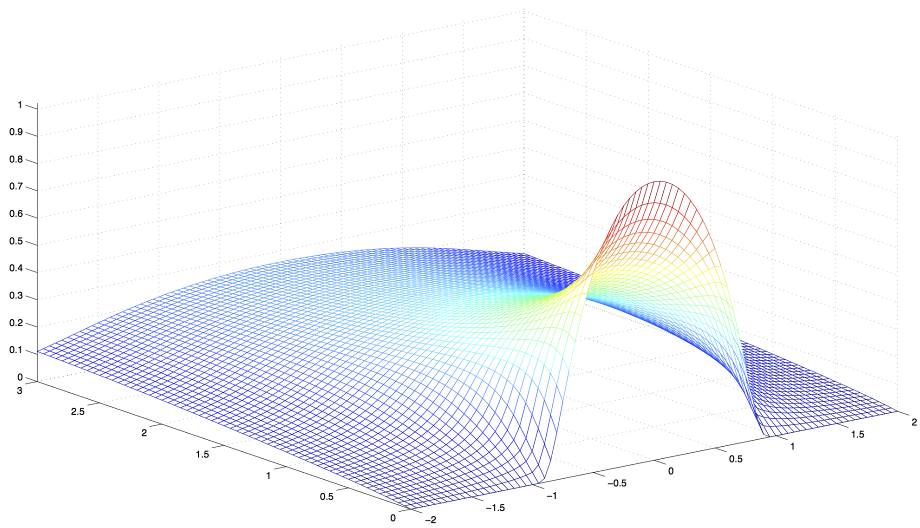

FIGURE 1.13: Numerical solution for $m = 2$

In Figures 1.12 and 1.13 we show the numerical solutions for two small m where the expected fat tail is observed. In the Figure 1.14 we observe the typical very slow diffusion of the porous medium equation with high values of m. The expected mesa solution studied in [80] is clearly observed.

A case where the nonlinearity is not a power, is presented in Figure 1.15 as an example where the numerical method is used for more general φ.

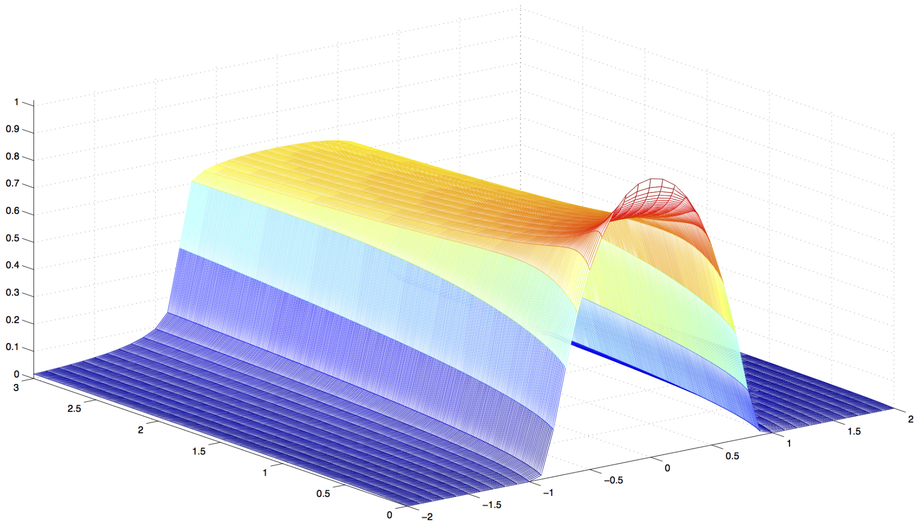

FIGURE 1.14: Numerical solution for $m = 10$

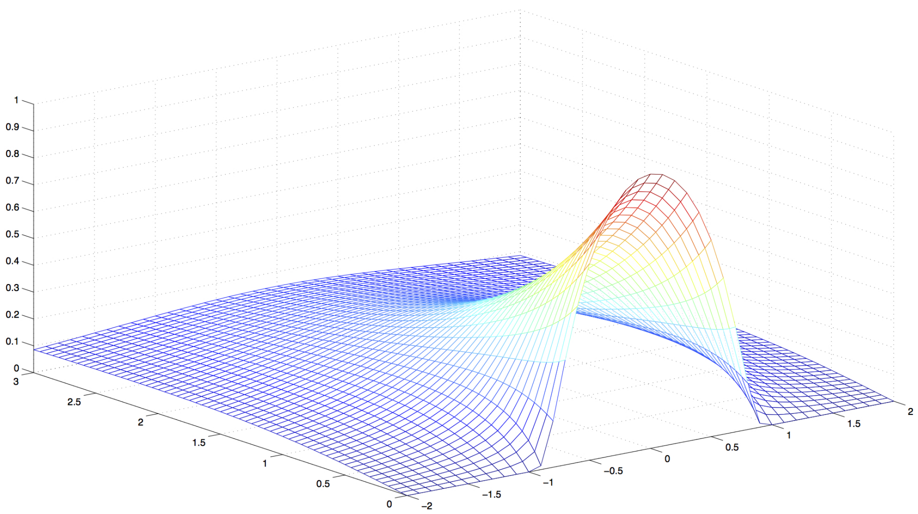

FIGURE 1.15: Numerical solution for $\varphi(u) = log(u + 1)$

1.7 Comments

1.7.1 Proofs for $N > 1$

As we commented before, all the results and proofs of this chapter can be very easily adapted for $N > 1$, but in order to convince the reader, let us at least formulate the numerical scheme in this case.

We need to introduce some multiindex notation for the spatial discretization, $i = (i_1, \dots, i_N)$ with $i_l = 0, \dots, I_l$ for $l = 1, \dots, N$ where I_l is the number of nodes of our mesh in the l-th dimensional direction. We will also use say that

$$1_l = (0, \dots, \underbrace{1}_{l-esim}, \dots, 0)$$

In this way, Problem (1.3.3)-(1.3.4) becomes

$$\begin{cases} \dfrac{\displaystyle\sum_{l=1}^{N} \left((W_j)_{i+1_l}^k + (W_j)_{i-1_l}^k\right) + (W_j)_i^{k+1} + (W_j)_i^{k-1} - 2(N+1)(W_j)_i^k}{\Delta x^2} = 0, & 0 < i < I, \\ & 0 < k < K, \\ (W_j)_i^0 = \left(\dfrac{\Delta t}{\Delta x}\left((W_{j-1})_i^1 - (W_{j-1})_i^0\right) + [(W_{j-1})_i^0]^{1/m}\right)^m, & 0 < i < I, \\ (W_j)_i^k = 0, & \text{otherwise}, \end{cases} \tag{1.7.1}$$

where the solution of

$$\begin{cases} \dfrac{\sum_{l=1}^{N} \left[(W_j)_{i+1_l}^k + (W_j)_{i-1_l}^k\right] + (W_0)_i^{k+1} + (W_0)_i^{k-1} - 2(N+1)(W_0)_i^k}{\Delta x^2} = 0, & 0 < i < I, \\ & 0 < k < K, \\ (W_0)_i^0 = f^m(x_i), & 0 < i < I, \\ (W_j)_i^k = 0, & \text{otherwise}, \end{cases}$$

is used to start the numerical method.

With this multi-index notation the proofs for the local truncation error, existence, uniqueness and convergence are valid and straight forward without any extra change.

1.7.2 Signed initial data

We have assume that the initial data f is a nonnegative function only for simplicity. One can easy observe that all the proofs are valid for f with sign with very small changes. Again, problems could come proving the required regularity for the solutions of (1.2.1).

Let us call $b_{min} = \min_x\{f^m(x), 0\}$. Then, in Theorem 1.3.2 the same argument holds to prove that $(U_j)_i^0 \geq b_{min}$. Then, the new maximum/minimum principle states that for all i, j, k we have have that $b_{min} \leq (U_j)_i^0 \leq b_{\max}$. In Theorem 1.3.5 nothing change since we are always talking about errors in absolute value.

Chapter 2

Finite and infinite speed of propagation for porous medium equations with fractional pressure

2.1 Introduction

In this chapter we study the following nonlocal evolution equation

$$\begin{cases} u_t(x,t) = \nabla\cdot(u^{m-1}\nabla p), \quad p = (-\Delta)^{-s}u, & \text{for } x\in\mathbb{R}^N,\ t>0,\\ u(x,0) = u_0(x) & \text{for } x\in\mathbb{R}^N, \end{cases} \tag{2.1.1}$$

for $m>1$ and $u(x,t)\geq 0$. The model formally resembles the classical *Porous Medium Equation* (PME)

$$u_t = \Delta u^m = \nabla(mu^{m-1}\nabla u)$$

where the pressure p depends linearly on the density function u according to the Darcy Law. In this model the pressure p takes into consideration nonlocal effects through the Inverse Fractional Laplacian operator $\mathcal{K}_s = (-\Delta)^{-s}$, that is, the Riesz potential of order $2s$. The problem is posed for $x\in\mathbb{R}^N$, $N\geq 1$ and $t>0$. The initial data $u_0:\mathbb{R}^N\to[0,\infty)$ is bounded with compact support or fast decay at infinity.

As a motivating precedent, in the work [25] Caffarelli and Vázquez proposed the following model of porous medium equation with nonlocal diffusion effects

$$\partial_t u = \nabla\cdot(u\nabla p), \quad p = (-\Delta)^{-s}u. \tag{2.1.2}$$

The study of this model has been performed in a series of papers as follows. In [25], Caffarelli and Vázquez developed the theory of existence of bounded weak solutions that propagate with finite speed. In [24], the same authors proved the asymptotic time behavior of the solutions. Self-similar non-negative solutions are obtained by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density, called obstacle Barenblatt solutions. Finally, in [22], Caffarelli, Soria and Vázquez considered the regularity and the $L^1 - L^\infty$ smoothing effect. The regularity for $s = 1/2$ has been recently done in [23]. The study of fine asymptotic behavior (rates of convergence) for (2.1.2) has been performed by Carrillo, Huang, Santos and Vázquez [27] in the one dimensional setting. Putting $m = 2$ in (2.1.1), we recover Problem (2.1.2).

A main question in this kind of nonlocal nonlinear diffusion models is to decide whether compactly supported data produce compactly supported solutions, a property known as finite speed of propagation. Surprisingly, the answer was proved to be positive for $m = 2$ in paper [25], for $m = 1$ we get the linear fractional heat equation, that is explicitly solvable by convolution with a positive kernel, hence it has infinite speed of propagation. The main motivation of this chapter is establishing the alternative finite/infinite speed of propagation for the solutions of Problem (2.1.1) depending on the parameter m. In the process we construct a theory of existence of solutions and derive the main properties. A modification of the numerical methods developed in [40, 41] pointed to us to the possibility of having two different propagation properties.

Other related models. Equation (2.1.2) with $s = 1/2$ in dimension $N = 1$ has been proposed by Head [49] to describe the dynamics of dislocation in crystals. The model is written in the integrated form as

$$v_t + |v_x|(-\partial^2/\partial_{xx})^{1/2}v = 0.$$

The dislocation density is $u = v_x$. This model has been recently studied by Biler, Karch and Monneau in [13], where they prove that the problem enjoys the properties of uniqueness and comparison of viscosity solutions. The relation between u and v is very interesting and will be used by us in the final sections.

Another possible generalization of the (2.1.2) model is

$$\partial_t u = \nabla \cdot (u\nabla p), \quad p = (-\Delta)^{-s}(|u|^{m-2}u),$$

that has been investigated by Biler, Imbert and Karch in [11, 12]. They prove the existence of weak solutions and they find explicit self-similar solutions with compact support for all $m > 1$. The finite speed of propagation for every weak solution has been done in [56].

The second nonlocal version of the classical PME is the model

$$u_t = -(-\Delta)^{s'} u^{m'}, \quad m' > 0,$$

known as the *Fractional Porous Medium Equation* (FPME). This model has infinite speed of propagation and the existence of fundamental solutions of self-similar type or Barenblatt solutions is known for $m > (N-2s')_+/N$. We refer to the recent works [17, 37, 38, 79]. The (FPME) model for $m' = 1$, also called linear fractional Heat Equation, coincides with model (2.1.1) for $s = 1 - s'$, $m = 1$. The linear model has been recently considered in [10], where several definitions of solutions adapted to the nonlocal character of the model are discussed.

2.1.1 Main results

We first propose a definition of solution and establish the existence and main properties of the solutions.

Definition 2.1.1. Let $m > 1$. We say that u is a weak solution of (2.1.1) in $Q_T = \mathbb{R}^N \times (0,T)$ with nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$ if (i) $u \in L^1(Q_T)$, (ii) $\nabla \mathcal{K}_s[u] \in L^1([0,T) : L^1_{loc}(\mathbb{R}^N))$, (iii) $u^{m-1}\nabla \mathcal{K}_s[u] \in L^1(Q_T)$, and (iv)

$$\int_0^T \int_{\mathbb{R}^N} u\phi_t dxdt - \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla \mathcal{K}_s[u] \nabla \phi dxdt + \int_{\mathbb{R}^N} u_0(x)\phi(x,0)dx = 0 \qquad (2.1.3)$$

holds for every test function ϕ in Q_T such that $\nabla\phi$ is continuous, ϕ has compact support in $\mathbb{R}^N$ for all $t \in (0,T)$ and vanishes near $t = T$.

Before entering the discussion of finite versus infinite propagation, we study the question of existence. We have the following result for $1 < m < 2$.

Theorem 2.1.2. *Let $m \in (1,2)$, $N \geq 1$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a weak solution u of equation (2.1.1) with initial data u_0 such that $u \in L^1(Q_T) \cap L^\infty(Q_T)$ and $\nabla \mathcal{H}_s[u] \in L^2(Q_T)$. Moreover, u has the following properties:*

1. ***(Conservation of mass)*** *For all $t > 0$ we have $\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$.*
2. ***(L^∞ estimate)*** *For all $t > 0$ we have $||u(\cdot,t)||_\infty \leq ||u_0||_\infty$.*
3. ***(First Energy estimate)*** *For all $t > 0$,*

$$C \int_0^t \int_{\mathbb{R}^N} |\nabla \mathcal{H}_s[u]|^2 dxdt + \int_{\mathbb{R}^N} u(t)^{3-m} dx \leq \int_{\mathbb{R}^N} u_0^{3-m} dx,$$

with $C = (2-m)(3-m) > 0$.

4. ***(Second Energy estimate)*** *For all $t > 0$,*

$$\frac{1}{2}\int_{\mathbb{R}^N} |\mathcal{H}_s[u]|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} |\nabla \mathcal{K}_s[u]|^2 dx \le \frac{1}{2}\int_{\mathbb{R}^N} |\mathcal{H}_s[\widehat{u}_0]|^2 dx.$$

The existence for $m \ge 2$ is covered in the following result.

Theorem 2.1.3. *Let $m \in [2,3)$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be such that*

$$0 \le u_0(x) \le Ae^{-a|x|} \text{ for some } A, a > 0. \tag{2.1.4}$$

Then there exists a weak solution u of equation (2.1.1) with initial data u_0 such that $u \in L^1(Q_T) \cap L^\infty(Q_T)$, $\nabla \mathcal{H}_s[u] \in L^2(Q_T)$ and u satisfies the properties 1, 2, 4 of Theorem 2.1.2. Moreover, the solution decays exponentially in $|x|$ and the first energy estimate holds in the form

$$|C| \int_0^t \int_{\mathbb{R}^N} |\nabla \mathcal{H}_s[u]|^2 dx dt + \int_{\mathbb{R}^N} u_0^{3-m} dx \le \int_{\mathbb{R}^N} u(t)^{3-m} dx$$

where $C = C(m) = (2-m)(3-m)$.

We should have covered existence in the whole range $m \ge 2$ where we want to prove finite speed of propagation for the constructed weak solutions, see Theorem 2.1.5. But the existence theory used in the previous theorem breaks down because of the negative exponents $3-m$ that would appear in the first energy estimate for $m > 3$ (a logarithm would appear for $m = 3$). A new existence approach avoiding such estimate is needed, and this can be done but is not immediate. We have refrained from presenting such a study here because it would divert us too much from the main interest.

Remark 2.1.4. (i) The proofs of Theorems 2.1.2 and 2.1.3 provide a class of weak solutions obtained as the limit of an approximation process. We refer to equation ($P_{\epsilon\delta\mu R}$) and Section 2.6. We will call them **constructed weak solutions**.

(ii) We point out that part (a) of Theorem 2.1.5 would still be true when $m \ge 3$ once we supply an existence theory for constructed weak solutions of this kind.

The following is our most important contribution, which deals with the property of finite propagation of the solutions depending on the value of m.

Theorem 2.1.5. *a) Let $N \ge 1$, $m \in [2,3)$, $s \in (0,1)$ and let u be a constructed weak solution to problem (2.1.1) as in Theorem 2.1.3 with compactly supported initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, $u(\cdot,t)$ is also compactly supported for any $t > 0$, i.e. the solution has **finite speed of propagation**.*

b) Let $N = 1$, $m \in (1,2)$, $s \in (0,1)$ and let u be a constructed solution as in Theorem 2.1.2. Then for any $t > 0$ and any $R > 0$, the set $\mathcal{M}_{R,t} = \{x : |x| \geq R,\ u(x,t) > 0\}$ has positive measure even if u_0 is compactly supported. This is a weak form of ***infinite speed of propagation****. If moreover u_0 is radially symmetric and monotone non-increasing in $|x|$, then we get a clearer result: $u(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.*

Organization of the proofs

• In Section 2.3 we derive useful energy estimates valid for all $m > 1$. Due to the differences in the computations, we will separate the cases $m \neq 2, 3$ and $m = 3$.

• In Section 2.4, 2.5 and 2.6 we prove the existence of a weak solution of Problem (2.1.1) as the limit of a sequence of solutions to suitable approximate problems. The range of exponents is $1 < m < 3$.

• Section 2.7 deals with the property of finite speed of propagation for $m \geq 2$. See Theorem 2.7.1.

• In Section 2.8 we prove the infinite speed of propagation for $m \in (1,2)$ in the one-dimensional case. This section introduces completely different tools. Indeed, we develop a theory of viscosity solutions for the integrated equation $v_t + |v_x|^{m-1}(-\Delta)^{1-s}v = 0$, where $v_x = u$ the solution of (2.1.1), and we prove infinite speed of propagation in the usual sense for the solution v of the integrated problem.

Though we do not get the same type of infinite propagation result for $1 < m < 2$ in several spatial dimensions, the evidence (partial results and explicit solutions) points in that direction, see the comments in Section 2.10.

2.2 Functional setting

We will work with the following functional spaces (see [42]). Let $s \in (0,1)$. Let $\mathcal{F}$ denote the Fourier transform. We consider

$$H^s(\mathbb{R}^N) = \left\{ u : L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s})|\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\}$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi.$$

For functions $u \in H^s(\mathbb{R}^N)$, the Fractional Laplacian is defined by

$$(-\Delta)^s u(x) = C_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy = C\mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)),$$

where $C_{N,s} = \pi^{-(2s+N/2)}\Gamma(N/2+s)/\Gamma(-s)$. Then

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + C\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}.$$

For functions u that are defined on a subset $\Omega \subset \mathbb{R}^N$ with $u = 0$ on the boundary $\partial\Omega$, the fractional Laplacian and the $H^s(\mathbb{R}^N)$ norm are computed by extending the function u to all $\mathbb{R}^N$ with $u = 0$ in $\mathbb{R}^N \setminus \Omega$. For technical reasons we will only consider the case $s < 1/2$ in $N = 1$ dimensional space.

The inverse operator $(-\Delta)^{-s}$ coincides with the Riesz potential of order $2s$ that will be denoted here by $\mathcal{K}_s$. It can be represented by convolution with the Riesz kernel K_s:

$$\mathcal{K}_s[u] = K_s * u, \quad K_s(x) = \frac{1}{c(N,s)}|x|^{-(N-2s)},$$

where $c(N,s) = \pi^{N/2-2s}\Gamma(s)/\Gamma((N-2s)/2)$. The Riesz potential $\mathcal{K}_s$ is a self-adjoint operator. The square root of $\mathcal{K}_s$ is $\mathcal{K}_{s/2}$, i.e. the Riesz potential of order s (up to a constant). We will denote it by $\mathcal{H}_s := (\mathcal{K}_s)^{1/2}$. Then $\mathcal{H}_s$ can be represented by convolution with the kernel $K_{s/2}$. We will write $\mathcal{K}$ and $\mathcal{H}$ when s is fixed and known. We refer to [63] for the arguments of potential theory used throughout the chapter.

The inverse fractional Laplacian $\mathcal{K}_s[u]$ is well defined as an integral operator for all $s \in (0,1)$ in dimension $N \geq 2$, and $s \in (0,1/2]$ in the one-dimensional case $N = 1$. We extend our result to the remaining case $s \in (1/2,1)$ by giving a suitable meaning to the combined operator $(\nabla\mathcal{K}_s)$. The details concerning this case will be given in Section 2.6.5.

For functions depending on x and t, convolution is applied for fixed t with respect to the spatial variables and we then write $u(t) = u(\cdot,t)$.

2.2.1 Functional inequalities related to the fractional Laplacian

We recall some functional inequalities related to the fractional Laplacian operator that we used throughout the chapter. We refer to [38] for the proofs.

Lemma 2.2.1 (Stroock-Varopoulos Inequality). *Let $0 < s < 1$, $q > 1$. Then*

$$\int_{\mathbb{R}^N} |v|^{q-2}v(-\Delta)^s v dx \geq \frac{4(q-1)}{q^2}\int_{\mathbb{R}^N}\left|(-\Delta)^{s/2}|v|^{q/2}\right|^2 dx \tag{2.2.1}$$

for all $v \in L^q(\mathbb{R}^N)$ *such that* $(-\Delta)^s v \in L^q(\mathbb{R}^N)$.

Lemma 2.2.2 (**Generalized Stroock-Varopoulos Inequality**). *Let* $0 < s < 1$. *Then*

$$\int_{\mathbb{R}^N} \psi(v)(-\Delta)^s v dx \geq \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\Psi(v)\right|^2 dx \tag{2.2.2}$$

whenever $\psi' = (\Psi')^2$.

Theorem 2.2.3 (**Sobolev Inequality**). *Let* $0 < s < 1$ *(*$s < \frac{1}{2}$ *if* $N = 1$*). Then*

$$\|f\|_{\frac{2N}{N-2s}} \leq \mathcal{S}_s \left\|(-\Delta)^{s/2} f\right\|_2, \tag{2.2.3}$$

where the best constant is given in [17] page 31.

2.2.2 Approximation of the Inverse Fractional Laplacian $(-\Delta)^{-s}$

We consider an approximation $\mathcal{K}_s^\epsilon$ as follows. Let $K_s(z) = c_{N,s}|z|^{-(N-2s)}$ the kernel of the Riesz potential $\mathcal{K}_s = (-\Delta)^{-s}$, $0 < s < 1$ ($0 < s < 1/2$ if $N = 1$). Let $\rho_\epsilon(x) = \epsilon^{-N}\rho(x/\epsilon)$, $\epsilon > 0$ a standard mollifying sequence, where ρ is positive, radially symmetric and decreasing, $\rho \in C_c^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \rho \, dx = 1$. We define the regularization of K_s as $K_s^\epsilon = \rho_\epsilon \star K_s$. Then

$$\mathcal{K}_s^\epsilon[u] = K_s^\epsilon \star u \tag{2.2.4}$$

is an approximation of the Riesz potential $\mathcal{K}_s = (-\Delta)^{-s}$. Moreover, $\mathcal{K}_s$ and $\mathcal{K}_s^\epsilon$ are self-adjoint operators with $\mathcal{K}_s = (\mathcal{H}_s)^2$, $\mathcal{K}_s^\epsilon = (\mathcal{H}_s^\epsilon)^2$. Also, $\rho = \sigma * \sigma$ where σ has the same properties as ρ. Then, we can write $\mathcal{H}_s^\epsilon$ as the operator with kernel $K_{s/2} * \sigma_\epsilon$. That is:

$$\int_{\mathbb{R}^N} u \ \mathcal{K}_s^\epsilon[u] dx = \int_{\mathbb{R}^N} |\mathcal{H}_s^\epsilon[u]|^2 dx.$$

Also $\mathcal{H}_s^\epsilon$ commutes with the gradient:

$$\nabla \mathcal{H}_s^\epsilon[u] = \mathcal{H}_s^\epsilon[\nabla u].$$

2.3 Basic estimates

In what follows, we perform formal computations on the solution of Problem (2.1.1), for which we assume smoothness, integrability and fast decay as $|x| \to \infty$. The useful computations for the theory of existence and propagation will be justified later by the approximation process. We fix $s \in (0, 1)$ and $m \geq 1$. Let u be the solution of Problem (2.1.1) with initial data $u_0 \geq 0$. We assume $u \geq 0$ for the beginning. This property will be proved later.

• **Conservation of mass:**

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_t dx = \int_{\mathbb{R}^N} \nabla\cdot(u^{m-1}\nabla\mathcal{K}_s[u])dx = 0. \tag{2.3.1}$$

• **First energy estimate:** The estimates here are significantly different depending on the exponent m. Therefore, we consider the cases:

CASE $m = 3$:

$$\frac{d}{dt}\int_{\mathbb{R}^N} \log u(x,t)dx \;=\; \int_{\mathbb{R}^N} \frac{u_t}{u}dx = \int_{\mathbb{R}^N} \nabla u\cdot\nabla\mathcal{K}_s[u] = \int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx.$$

Therefore, by the conservation of mass (2.3.1) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N} (u-\log u)dx = -\int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx. \tag{2.3.2}$$

CASE $m \neq 3$:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}(x,t)dx &= (3-m)\int_{\mathbb{R}^N} u^{2-m}u_t dx = (3-m)\int_{\mathbb{R}^N} u^{2-m}\nabla(u^{m-1}\nabla\mathcal{K}_s[u])dx\\ &= -(3-m)(2-m)\int_{\mathbb{R}^N} \nabla u\cdot\nabla\mathcal{K}_s[u]dx = -C\int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx.\end{aligned}$$

Here $C = (3-m)(2-m)$ is negative for $m \in (2,3)$ and positive otherwise.

If $m > 3$ or $1 < m < 2$ then

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}dx = -|C|\int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx.$$

If $2 < m < 3$ then

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}dx = |C|\int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx,$$

or equivalently

$$\frac{d}{dt}\int_{\mathbb{R}^N} u - u^{3-m}dx = -|C|\int_{\mathbb{R}^N} |\nabla\mathcal{H}_s[u]|^2 dx.$$

• **Second energy estimate:**

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |\mathcal{H}_s[u](x,t)|^2 dx &= \int_{\mathbb{R}^N} \mathcal{H}_s[u](\mathcal{H}_s[u])_t dx = \int_{\mathbb{R}^N} \mathcal{K}_s[u]u_t dx\\ &= \int_{\mathbb{R}^N} \mathcal{K}_s[u]\nabla\cdot(u^{m-1}\nabla\mathcal{K}_s[u])dx = -\int_{\mathbb{R}^N} u^{m-1}|\nabla\mathcal{K}_s[u]|^2 dx.\end{aligned} \tag{2.3.3}$$

• **L^∞ estimate:** We prove that the $L^\infty(\mathbb{R}^N)$ norm does not increase in time. Indeed, at a point of maximum x_0 of u at time $t = t_0$, we have

$$u_t = (m-1)u^{m-1}\nabla u \cdot \nabla p + u^{m-1}\Delta \mathcal{K}_s[u].$$

The first term is zero since $\nabla u(x_0, t_0) = 0$. For the second one we have $-\Delta \mathcal{K}_s = (-\Delta)(-\Delta)^{-s} = (-\Delta)^{1-s}$ so that

$$\Delta \mathcal{K}_s[u](x_0,t_0) = -(-\Delta)^{1-s}u(x_0,t_0) = -c\int_{\mathbb{R}^N} \frac{u(x_0,t_0)-u(y,t_0)}{|x_0-y|^{N-2(1-s)}}dy \le 0,$$

where $c = c(s, N) > 0$. We conclude by the positivity of u that

$$u_t(x_0,t_0) = u^{m-1}(x_0,t_0)\Delta \mathcal{K}_s[u](x_0,t_0) \le 0.$$

• **Conservation of positivity:** we prove that if $u_0 \ge 0$ then $u(t) \ge 0$ for all times. The argument is similar to the one above.

• **L^p estimates for $1 < p < \infty$.** The following computations are valid for all $m \ge 1$, since $p + m - 2 > 0$:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} u^p(x,t)dx = p\int_{\mathbb{R}^N} u^{p-1}\nabla\cdot(u^{m-1}\nabla \mathcal{K}_s[u])dx\\
= &-p\int_{\mathbb{R}^N} u^{m-1}\nabla(u^{p-1})\cdot\nabla \mathcal{K}_s[u]dx = -\frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N}\nabla(u^{p+m-2})\cdot\nabla\mathcal{K}_s[u]dx\\
= &\frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N} u^{p+m-2}\Delta\mathcal{K}_s[u]dx = -\frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N} u^{p+m-2}(-\Delta)^{1-s}u\ dx\\
\le &-\frac{4p(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{1-s}{2}}u^{\frac{m+p-1}{2}}\right|^2 dx,
\end{aligned}$$

where we applied the Stroock-Varopoulos inequality (2.2.1) with $r = m + p + 1$. We obtain that $\int_{\mathbb{R}^N} u^p(t)dx$ is non-increasing in time. Moreover, by Sobolev Inequality (2.2.3) applied to the function $f = u^{(m+p-1)/2}$, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^p(x,t)dx \le -\frac{4p(p-1)}{(m+p-1)^2\mathcal{S}_{1-s}^2}\left(\int_{\mathbb{R}^N}|u(x,t)|^{\frac{N(m+p-1)}{N-2+2s}}dx\right)^{\frac{N-2+2s}{N}},$$

with the restriction of $s > 1/2$ if $N = 1$.

2.4 Existence of smooth approximate solutions for $m \in (1, \infty)$

Our aim is to solve the initial-value problem (2.1.1) posed in $Q = \mathbb{R}^N \times (0, \infty)$ or at least $Q_T = \mathbb{R}^N \times (0, T)$, with parameter $0 < s < 1$. We will consider initial data $u_0 \in L^1(\mathbb{R}^N)$. We assume for technical reasons that u_0 is bounded and we also impose decay conditions as $|x| \to \infty$.

2.4.1 Approximate problem

We make an approach to problem (2.1.1) based on regularization, elimination of the degeneracy and reduction of the spatial domain. Once we have solved the approximate problems, we derive estimates that allow us to pass to the limit in all the steps one by one, to finally obtain the existence of a weak solution of the original problem (2.1.1). Specifically, for small $\epsilon, \delta, \mu \in (0, 1)$ and $R > 0$ we consider the following initial boundary value problem posed in $Q_{T,R} = \{x \in B_R(0),\ t \in (0, T)\}$

$$\begin{cases} (U_1)_t = \delta \Delta U_1 + \nabla \cdot (d_\mu(U_1) \nabla \mathcal{K}_s^\epsilon[U_1]) & \text{for } (x,t) \in Q_{T,R}, \\ U_1(x,0) = \widehat{u}_0(x) & \text{for } x \in B_R(0), \\ U_1(x,t) = 0 & \text{for } x \in \partial B_R(0),\ t \geq 0. \end{cases} \qquad (P_{\epsilon\delta\mu R})$$

The regularization tools that we use are as follows. $\widehat{u}_0 = \widehat{u}_{0,\epsilon,R}$ is a nonnegative, smooth and bounded approximation of the initial data u_0 such that $\|\widehat{u}_0\|_\infty \leq \|u_0\|_\infty$ for all $\epsilon > 0$. For every $\mu > 0$, $d_\mu : [0, \infty) \to [0, \infty)$ is a continuous function defined by

$$d_\mu(v) = (v + \mu)^{m-1}. \qquad (2.4.1)$$

The approximation of $\mathcal{K}_s^\epsilon$ of $\mathcal{K}_s = (-\Delta)^{-s}$ is made as before in Section 2.2. The existence of a solution $U_1(x,t)$ to Problem ($P_{\epsilon\delta\mu R}$) can be done by classical methods and the solution is smooth. See for instance [64] for similar arguments.

In the weak formulation we have

$$\int_0^T \int_{B_R} U_1(\phi_t - \delta \Delta \phi) dx dt - \int_0^T \int_{B_R} d_\mu(U_1) \nabla \mathcal{K}_s^\epsilon[U_1] \nabla \phi dx dt + \int_{B_R} \widehat{u}_0(x) \phi(x, 0) dx = 0 \qquad (2.4.2)$$

valid for smooth test functions ϕ that vanish on the spatial boundary ∂B_R and for large t. We use the notation $B_R = B_R(0)$.

Notations. The existence of a weak solution of problem (2.1.1) is done by passing to the limit step-by-step in the approximating problems as follows. We denote by U_1

the solution of the approximating problem ($P_{\epsilon\delta\mu R}$) with parameters ϵ, δ, μ, R. Then we will obtain $U_2(x,t) = \lim_{\epsilon\to 0} U_1(x,t)$. Thus U_2 will solve an approximating problem ($P_{\delta\mu R}$) with parameters δ, μ, R. Next, we take $U_3 = \lim_{R\to\infty} U_2(x)$ that will be a solution of Problem ($P_{\mu\delta}$), $U_4 := \lim_{\mu\to 0} U_3(x,t)$ solving Problem (P_δ). Finally we obtain $u(x,t) = \lim_{\delta\to 0} U_4(x,t)$ which solves problem (2.1.1).

2.4.2 A-priori estimates for the approximate problem

We derive suitable a-priori estimates for the solution $U_1(x,t)$ to Problem ($P_{\epsilon\delta\mu R}$) depending on the parameters ϵ, δ, μ, R.

• **Decay of total mass:** Since $U_1 \geq 0$ and $U_1 = 0$ in ∂B_R, then $\dfrac{\partial U_1}{\partial n} \leq 0$ and so, an easy computation gives us

$$\begin{aligned}\frac{d}{dt}\int_{B_R} U_1(x,t)dx &= \delta \int_{B_R} \Delta U_1 \, dx + \int_{B_R} \nabla\cdot(d_\mu(U_1)\nabla\mathcal{K}_\epsilon[U_1])dx \\ &= \int_{\partial B_R} \frac{\partial U_1}{\partial n} d\sigma + \int_{\partial B_R} d_\mu(U_1)\frac{\partial(\mathcal{K}_\epsilon[U_1])}{\partial n} d\sigma \leq 0. \qquad (2.4.3)\end{aligned}$$

We conclude that

$$\int_{B_R} U_1(x,t)dx \leq \int_{B_R} \widehat{u}_0(x) \quad \text{for all } t>0.$$

• **Conservation of L^∞ bound**: we prove that $0 \leq U_1(x,t) \leq ||\widehat{u}_0||_\infty$. The argument is as in the previous section, using also that at a minimum point $\Delta U_1 \geq 0$ and at a maximum point $\Delta U_1 \leq 0$. Also at this kind of points we have that

$$\nabla d_\mu(U_1) = d'_\mu(U_1)\nabla U_1 = 0.$$

• **Conservation of non-negativity**: $U_1(x,t) \geq 0$ for all $t>0$, $x \in B_R$. The proof is similar to the one in the previous section.

2.4.2.1 First energy estimate

We choose a function F_μ such that

$$F_\mu(0) = F'_\mu(0) = 0 \quad \text{and} \quad F''_\mu(u) = 1/d_\mu(u).$$

Then, with these conditions one can see that $F_\mu(z) > 0$ for all $z > 0$. Also $F_\mu(U_1)$ and $F'_\mu(U_1)$ vanish on $\partial B_r \times [0,T]$, therefore, after integrating by parts, we get

$$\frac{d}{dt}\int_{B_R} F_\mu(U_1)dx = -\delta\int_{B_R}\frac{|\nabla U_1|^2}{d_\mu(u)}dx - \int_{B_R}|\nabla \mathcal{H}_s^\epsilon[U_1]|^2dx, \tag{2.4.4}$$

where $\mathcal{H}_\epsilon = \mathcal{K}_\epsilon^{1/2}$. This formula implies that for all $0 < t < T$ we have

$$\int_{B_R} F_\mu(U_1(t))dx + \delta\int_0^t\int_{B_R}\frac{|\nabla U_1|^2}{d_\mu(U_1)}dxdt + \int_0^t\int_{B_R}|\nabla \mathcal{H}_s^\epsilon[U_1]|^2dxdt = \int_{B_R}F_\mu(\widehat{u}_0)dx. \tag{2.4.5}$$

This implies estimates for $|\nabla \mathcal{H}_s^\epsilon(U_1)|^2$ and $\delta|\nabla U_1|^2/d_\mu(U_1)$ in $L^1(Q_{T,R})$. We show how the upper bounds for such norms depend on the parameters ϵ, δ, R, μ and T.

The explicit formula for F_μ is as follows:

$$F_\mu(U_1) = \begin{cases} \dfrac{1}{(2-m)(3-m)}[(U_1+\mu)^{3-m} - \mu^{3-m}] - \dfrac{1}{2-m}\mu^{2-m}U_1 & \text{for } m \neq 2,3, \\ -\log(1+(U_1/\mu)) + U_1/\mu, & \text{for } m = 3, \\ (U_1+\mu)\log(1+(U_1/\mu)) - U_1, & \text{for } m = 2. \end{cases}$$

From formula (2.4.4) we obtain that the quantity $\int_{B_R} F_\mu(U_1(x,t))dx$ is non-increasing in time:

$$0 \leq \int_{B_R} F_\mu(U_1(x,t))dx \leq \int_{B_R} F_\mu(\widehat{u}_0)dx, \quad \forall t > 0.$$

Then, if we control the term $\int_{B_R} F_\mu(\widehat{u}_0)dx$, we will obtain uniform estimates independent of time $t > 0$ for the quantity

$$\delta\int_0^t\int_{B_R}\frac{|\nabla U_1|^2}{d_\mu(U_1)}dxdt + \int_0^t\int_{B_R}|\nabla \mathcal{H}_s^\epsilon[U_1]|^2dxdt.$$

These estimate are different depending on the range of parameters m.

• **Uniform bound in the case** $m \in (1,2)$. We obtain uniform bounds in all parameters ϵ, R, δ, μ for the energy estimate (2.4.5), that allow us to pass to the limit and obtain a solution of the original problem (2.1.1). By the Mean Value Theorem

$$\begin{aligned}\int_{B_R} F_\mu(\widehat{u}_0)dx &\leq \frac{1}{(2-m)(3-m)}\int_{B_R}[(\widehat{u}_0+\mu)^{3-m} - \mu^{3-m}]dx \\ &\leq \frac{1}{2-m}\int_{B_R}(\widehat{u}_0+\mu)^{2-m}\widehat{u}_0dx \\ &\leq \frac{1}{2-m}(\|u_0\|_\infty + 1)^{2-m}\int_{\mathbb{R}^N} u_0dx.\end{aligned}$$

Our main estimate in the case $m \in (1,2)$ is:

$$\delta \int_0^t \int_{B_R} \frac{|\nabla U_1|^2}{d_\mu(U_1)} dxdt + \int_0^t \int_{B_R} |\nabla \mathcal{H}_s^\epsilon[U_1]|^2 dxdt \leq C_1, \tag{2.4.6}$$

where $C_1 = C_1(m, u_0) = \dfrac{2}{(2-m)}(\|u_0\|_\infty + 1)^{2-m} \|u_0\|_{L^1(\mathbb{R}^N)}$. This is a bound independent of the parameters ϵ, δ, R and μ.

- **Upper bound in the case** $m \in (2,3)$.

$$\int_{B_R} F_\mu(\widehat{u}_0)dx = -\frac{1}{(m-2)(3-m)} \int_{B_R} [(\widehat{u}_0 + \mu)^{3-m} - \mu^{3-m}]dx + \frac{1}{m-2}\mu^{2-m} \int_{B_R} \widehat{u}_0 dx$$
$$\leq \frac{1}{m-2}\mu^{2-m} \int_{B_R} \widehat{u}_0 dx \leq \frac{1}{m-2}\mu^{2-m} \int_{\mathbb{R}^N} u_0 dx.$$

This upper bound will allow us to obtain compactness arguments in ϵ and R for fixed μ. We will be able to control $\int_{B_R} F_\mu(\widehat{u}_0)dx - \int_{B_R} F_\mu(U_1(t))dx$ uniformly in μ, after passing to the limit as $\epsilon \to 0$ and $R \to \infty$, due to a exponential decay result on the solution at time $t \in [0,T]$ that we will prove in Section 2.5 and the conservation of mass.

Remark. These techniques do not apply in the case $m \geq 3$ because even an exponential decay on the solution is not enough to control the terms in the first energy estimate.

2.4.2.2 Second energy estimate

Similar computations to (2.3.3) yields to the following energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_{B_R} |\mathcal{H}_s^\epsilon[U_1]|^2 dx \leq -\delta \int_{B_R} |\nabla \mathcal{H}_s^\epsilon[U_1]|^2 dx - \int_{B_R} (U_1+\mu)^{m-1} |\nabla \mathcal{K}_s^\epsilon[U_1]|^2 dx.$$

This implies that, for all $0 < t < T$ we have

$$\frac{1}{2}\int_{B_R} |\mathcal{H}_s^\epsilon[U_1(t)]|^2 dx + \delta \int_0^T \int_{B_R} |\nabla \mathcal{H}_s^\epsilon[U_1]|^2 dx + \int_0^T \int_{B_R} (U_1+\mu)^{m-1} |\nabla \mathcal{K}_s^\epsilon[U_1]|^2 dx \leq \frac{1}{2}\int_{B_R} |\mathcal{H}_s^\epsilon[\widehat{u}_0]|^2 dx. \tag{2.4.7}$$

Note that the last integral is well defined as long as $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

2.5 Exponential tail control in the case $m \geq 2$

In this section and the next one we will give the proof of Theorem 2.1.3. Weak solutions of the original problem are constructed by passing to the limit after a tail control step.

We develop a comparison method with a suitable family of barrier functions, that in [25] received the name of *true supersolutions*.

Theorem 2.5.1. *Let $0 < s < 1/2$, $m \geq 2$ and let U_1 be the solution of Problem ($P_{\epsilon\delta\mu R}$). We assume that U_1 is bounded $0 \leq U_1(x,t) \leq L$ and that u_0 lies below a function of the form*

$$V_0(x) = Ae^{-a|x|}, \quad A, a > 0.$$

If A is large, then there is a constant $C > 0$ that depends only on (N, s, a, L, A) such that for any $T > 0$ we will have the comparison

$$U_1(x,t) \leq Ae^{Ct-a|x|} \quad \textit{for all} \quad x \in \mathbb{R}^N,\ 0 < t \leq T.$$

Proof. • **Reduction.** By scaling we may put $a = L = 1$. This is done by considering instead of U_1, the function $\tilde{U_1}$ defined as

$$U_1(x,t) = L\tilde{U_1}(ax, bt), \quad b = L^{m-1}a^{2-2s}, \tag{2.5.1}$$

which satisfies the equation

$$(\tilde{U_1})_t = \delta_1 \Delta \tilde{U_1} + \nabla.(d_{\frac{\mu}{L}}(\tilde{U_1})\nabla \mathcal{K}_s^{\epsilon a}(\tilde{U_1})),$$

with $\delta_1 = a^{2s}\delta/L^{m-1}$. Note that then $\tilde{U_1}(x,0) \leq A_1 e^{-|x|}$ with $A_1 = A/L$. The corresponding bound for $\tilde{U_1}(x,t)$ will be $\tilde{U_1}(x,t) \leq A/L\ e^{C_1 t-|x|}$ with $C_1 = C/b = C\left(L^{m-1}a^{2-2s}\right)^{-1}$.

• **Contact analysis.** Therefore we assume that $0 \leq U_1(x,0) \leq 1$ and also that

$$U_1(x,0) \leq Ae^{-r}, \quad r = |x| > 0,$$

where $A > 0$ is a constant that will be chosen below, say larger than 2. Given constants C, ϵ and $\eta > 0$, we consider a radially symmetric candidate for the upper barrier function of the form

$$\widehat{U}(x,t) = Ae^{Ct-r} + hAe^{\eta t},$$

and we take h small. Then C will be determined in terms of A to satisfy a true supersolution condition which is obtained by contradiction at the first point (x_c, t_c) of possible contact of u and $\widehat{U}$.

The equation satisfied by u can be written in the form

$$(U_1)_t = \delta \Delta U_1 + (m-1)(u+\mu)^{m-2}\nabla U_1 \cdot \nabla p + (U_1+\mu)^{m-1}\Delta p, \qquad p = \mathcal{K}_s^{\epsilon}[U_1]. \tag{2.5.2}$$

We will obtain necessary conditions in order for equation (2.5.2) to hold at the contact point (x_c, t_c). Then, we prove there is a suitable choice of parameters C, A, η, h, μ such that the contact can not hold.

Estimates on u and p at the first contact point. For $0 < s < 1/2$, at the first contact point (x_c, t_c) we have the estimates

$$\partial_r U_1 = -Ae^{Ct_c - r_c}, \quad \Delta U_1 \leq Ae^{Ct_c - r_c}, \quad (U_1)_t \geq ACe^{Ct_c - r_c} + h\eta Ae^{\eta t_c}.$$

Since we assumed our solution u is bounded by $0 \leq u \leq 1$, then

$$U_1(x_c, t_c) = Ae^{Ct_c - r_c} + hAe^{\eta t_c} \leq 1. \tag{2.5.3}$$

Moreover, from [25] we have the following upper bounds for the pressure term at the contact point for $0 < s < 1/2$:

$$\Delta p(x_c, t_c) \leq K_1, \quad (-\partial_r p)(x_c, t_c) \leq K_2. \tag{2.5.4}$$

Note that we are considering a regularized version of the p used in [25]. Of course the estimates still true (maybe with slightly bigger constants) since U_1 is regular.

Necessary conditions at the first contact point. Equation (2.5.2) at the contact point (x_c, t_c) with $r_c = |x_c|$, implies that

$$\begin{aligned} ACe^{Ct_c - r_c} + h\eta Ae^{\eta t_c} \leq \delta Ae^{Ct_c - r_c} + (m-1)\left(U_1(x_c, t_c) + \mu\right)^{m-2}(-Ae^{Ct_c - r_c})(\partial_r p) + \\ + \left(U_1(x_c, t_c) + \mu\right)^{m-1} \Delta p. \end{aligned}$$

We denote $\xi := r_c + (\eta - C)t_c$. Using also (2.5.4) with $K = \max\{K_1, K_2\}$, we obtain, after we simplify the previous inequality by $Ae^{Ct_c - r_c}$,

$$C + h\eta e^{\xi} \leq \delta + (m-1)\left(U_1(x_c, t_c) + \mu\right)^{m-2} K + (U_1(x_c, t_c) + \mu)^{m-2}(1 + he^{\xi} + \frac{\mu}{A}e^{r_c - Ct_c})K,$$

and equivalently

$$C + \epsilon\eta e^{\xi} \leq \delta + K\left(u(x_c, t_c) + \mu\right)^{m-2}\left(m + he^{\xi} + \frac{\mu}{A}e^{r_c - Ct_c}\right).$$

We take $C = \eta$ and $\frac{\mu}{A} \leq h$. Then

$$C + hCe^{r_c} \leq \delta + K\left(U_1(x_c, t_c) + \mu\right)^{m-2}\left(m + he^{r_c} + he^{r_c - Ct_c}\right).$$

Moreover,

$$C + hCe^{r_c} \le \delta + K\left(U_1(x_c,t_c)+\mu\right)^{m-2}\left(m+2he^{r_c}\right).$$

By (2.5.3) we have that

$$\mu < U_1(x_c,t_c)+\mu < 1+\mu.$$

Since $m \ge 2$, then

$$C + hCe^{r_c} \le \delta + K\left(1+\mu\right)^{m-2}\left(m+2he^{r_c}\right).$$

This is impossible for C large enough such that

$$C \ge \delta + mK(1+\mu)^{m-2} \quad \text{and} \quad C \ge 2K\left(1+\mu\right)^{m-2}. \tag{2.5.5}$$

Since $\mu < 1$ and $\delta < 1$, then we can choose C as constant, only depending on m and K.

□

Theorem 2.5.2. *Let $1/2 \le s < 1$, $m \ge 2$. Under the assumptions of the previous theorem, the stated tail estimate works locally in time. The global statement must be replaced by the following: there exists an increasing function $C(t)$ such that*

$$u(x,t) \le Ae^{C(t)t-a|x|} \quad \textit{for all } x \in \mathbb{R}^N \textit{ and all } 0 \le t \le T. \tag{2.5.6}$$

Proof. The proof of this result is similar to the one in [25] but with a technical adaptation to our model. When $N \ge 2$, $1/2 \le s < 1$, the upper bound $\Delta p(x_c,t_c) \le K_0$ at the first contact point holds. Moreover, in [25], the following upper bound for $(-\partial_r p)(x_c,t_c)$ is obtained,

$$(-\partial_r p)(x_c,t_c) \le K_1 + K_2 ||U_1(t)||_1^{1/q} ||U_1(t)||_\infty^{(q-1)/q},$$

where $1 \le q < N/(2s-1)$. We know that $||U_1(t)||_\infty \le 1$ and before the first contact point we have that $U_1(x,t) \le Ae^{ct}e^{-|x|}$, hence $||U_1(t)||_1 \le K_3 Ae^{Ct}$. Therefore, if we consider $K = \max\{K_0, K_1, K_2 K_3^{1/q}\}$ we have that

$$\Delta p(x_c,t_c) \le K, \qquad (-\partial_r p)(x_c,t_c) \le K + KA^{1/q}e^{Ct_c/q}. \tag{2.5.7}$$

Using this estimates in the equation we obtain

$$\begin{aligned} C + h\eta e^{\xi} \le \delta + K(m-1)\left(u(x_c,t_c)+\mu\right)^{m-2}(1+A^{1/q}e^{Ct_c/q}) + \\ + K(u(x_c,t_c)+\mu)^{m-2}(1+he^{\xi}+\frac{\mu}{A}e^{r_c-Ct_c}). \end{aligned}$$

We put $C = \eta$, $h = \mu/A$ and use that $\mu < u(x_c, t_c) + \mu < 1 + \mu$ to get

$$C + hCe^{r_c} \le \delta + KA^{1/q}(m-1)\,(1+\mu)^{m-2}\,e^{Ct_c/q} + K(1+\mu)^{m-2}\,(m + 2he^{r_c})\,.$$

We consider $\mu < 1$. The contradiction argument works as before with the big difference that we must restrict the time so that $e^{Ct_c/q} \le 2$, which happens if

$$t_c \le T_1 = (q \log 2)/C.$$

Then

$$C + hCe^{r_c} \le \delta + 2^{m-1}KA^{1/q}(m-1) + 2^{m-2}Km + 2^{m-2}Khe^{r_c}.$$

Since $A > 1$ and $\delta < 1$, and hence $2^{m-1}KA^{1/q}(m-1) + 2^{m-2}Km < 2^{m-1}KA^{1/q}(2m-1)$, we get a contradiction by choosing C such that:

$$C = 2^m KA^{1/q} m \ge \delta + 2^{m-1}KA^{1/q}(2m-1).$$

We have proved that there will be no contact with the barrier

$$B_1(x,t) = Ae^{Ct-|x|}$$

for $t < T_1 = c_1 A^{-1/q}$ where $c_1 = \frac{q \log 2}{Km2^m}$.

We can repeat the argument for another time interval by considering the problem with initial value at time T_1, that is,

$$U_1(x, T_1) \le Ae^{CT_1-|x|} = A_1 e^{-|x|} \text{ where } A_1 = Ae^{CT_1},$$

and we get $U(x,t) \le e^{C_1 t-|x|}$ for $T_1 \le t < T_2 = c_1 A^{-1/q} e^{-CT_1/q}$ where $C_1 = Ce^{CT_1/q}$. In this way we could find an upper bound to a certain time for the solution depending on the initial data through A.

When $N = 1$, $1/2 \le s < 1$, the operator $\partial_r p$ and Δp are considered in the sense given in Section 2.6.5. $\square$

2.6 Existence of weak solutions for $m \in (1, 3)$

2.6.1 Limit as $\epsilon \to 0$

We begin with the limit as $\epsilon \to 0$ in order to obtain a solution of the equation

$$(U_2)_t = \delta \Delta U_2 + \nabla \cdot (d_\mu(U_2) \nabla \mathcal{K}_s[U_2]). \qquad (P_{\delta\mu R})$$

Let U_1 be the solution of ($P_{\epsilon\delta\mu R}$). We fix δ, μ and R and we argue for ϵ close to 0. Then, by the energy formula (2.4.6) and the estimates from Section 2.4.2 we obtain that

$$\delta \int_0^t \int_{B_R} \frac{|\nabla U_1|^2}{(U_1+\mu)^{m-1}} dxdt \le C(\mu, m, \widehat{u}_0), \quad \int_0^t \int_{B_R} |\nabla \mathcal{H}_s^\epsilon[U_1]|^2 \, dxdt \le C(\mu, m, \widehat{u}_0), \tag{2.6.1}$$

valid for all $\epsilon > 0$. Since $\|U_1\|_\infty \le \|u_0\|_\infty$ for all $\epsilon > 0$, then

$$\int_0^t \int_{B_R} |\nabla U_1|^2 dxdt \le C(\mu, m, u_0)(\|u_0\|_\infty + 1)^{m-1}, \quad \forall \epsilon > 0.$$

We recall that in the case $m \in (1,2)$ the constant C is independent of μ, that is $C = C(m, u_0)$.

I. Convergence as $\epsilon \to 0$. We perform an analysis of the family of approximate solutions $(U_1)_\epsilon$ in order to derive a compactness property in suitable functional spaces.

• Uniform boundedness: $U_1 \in L^\infty(Q_{T,R})$, and the bound $||U_1(t)||_{L^\infty(\mathbb{R}^N)} \le ||u_0||_{L^\infty(\mathbb{R}^N)}$ is independent of ϵ, δ, μ and R for all $t > 0$. Moreover $||U_1(t)||_{L^1(\mathbb{R}^N)} \le ||u_0||_{L^1(\mathbb{R}^N)}$ for all $t > 0$.

• Gradient estimates. From the energy formula (2.6.1) we derive

$$U_1 \in L^2([0,T] : H_0^1(B_R)), \quad \nabla \mathcal{H}_s^\epsilon[U_1] \in L^2([0,T] : L^2(\mathbb{R}^N))$$

uniformly bounded for $\epsilon > 0$. Since $\nabla \mathcal{H}_s^\epsilon[U_1]$ is "a derivative of order $1-s$ of U_1", we conclude that

$$U_1 \in L^2([0,T], H^{1-s}(\mathbb{R}^N)). \tag{2.6.2}$$

• Estimates on the time derivative $(U_1)_t$: we use the equation ($P_{\epsilon\delta\mu R}$) to obtain that

$$(U_1)_t \in L^2([0,T] : H^{-1}(\mathbb{R}^N)) \tag{2.6.3}$$

as follows:

(a) Since $U_1 \in L^2([0,T] : H_0^1(B_R))$ we obtain that $\Delta U_1 \in L^2([0,T] : H^{-1}(\mathbb{R}^N))$.

(b) As a consequence of the Second Energy Estimate and the fact that $U_1 \in L^\infty(Q_T)$, we have that $d_\mu(U_1)\nabla \mathcal{K}_s^\epsilon[U_1] \in L^2([0,T] : L^2(\mathbb{R}^N))$, therefore $\nabla \cdot (d_\mu(U_1)\nabla \mathcal{K}_s[U_1])) \in L^2([0,T] : H^{-1}(\mathbb{R}^N))$.

Now, since $||(U_1)_t||_{L_t^1([0,T]:H^{-1+s}(\mathbb{R}^N))} \le T^{1/2}||(U_1)_t||_{L_t^2([0,T]:H^{-1+s}(\mathbb{R}^N))}$, expressions (2.6.2) and (2.6.3), allow us to apply the compactness criteria of Simon, see Lemma 2.9.3 in the

Appendix, in the context of

$$H^{1-s}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),$$

and we conclude that the family of approximate solutions (U_1) is relatively compact in $L^2([0,T] : L^2(\mathbb{R}^N))$. Therefore, there exists a limit $(U_1)_{\epsilon,\delta,\mu,R} \to (U_2)_{\delta,\mu,R}$ as $\epsilon \to 0$ in $L^2([0,T] : L^2(\mathbb{R}^N))$, up to subsequences. Note that, since $(U_1)_\epsilon$ is a family of positive functions defined on B_R and extended to 0 in $\mathbb{R}^N \setminus B_R$, then the limit $U_2 = 0$ a.e. on $\mathbb{R}^N \setminus B_R$. We obtain that

$$U_1 \stackrel{\epsilon\to 0}{\longrightarrow} U_2 \quad \text{in } L^2([0,T] : L^2(B_R)) = L^2(B_R \times [0,T]). \tag{2.6.4}$$

II. The limit is a solution of the new problem $(P_{\delta\mu R})$. More exactly, we pass to the limit as $\epsilon \to 0$ in the definition (2.4.2) of a weak solution of Problem $(P_{\epsilon\delta\mu R})$ and prove that the limit $U_2(x,t)$ of the solutions $U_1(x,t)$ is a solution of Problem $(P_{\delta\mu R})$. The convergence of the first integral in (2.4.2) is justified by (2.6.4) since

$$\left|\int_0^T \int_{B_R} (U_1 - U_2)(\phi_t - \delta\Delta\phi)dxdt\right| \leq ||U_1 - U_2||_{L^2(B_R\times[0,T])}||\phi_t - \delta\Delta\phi||_{L^2(B_R\times[0,T])}.$$

Convergence of the second integral in (2.4.2) is consequence of the second energy estimate (2.4.7) as we show now. First we note that

$$||(U_1+\mu)^{\frac{m-1}{2}}\nabla\mathcal{K}_s^\epsilon[U_1]||_{L^2(B_R\times(0,T))} \leq C$$

for some constant $C > 0$ independent of ϵ. Then, Banach-Alaoglu ensures that there exists a subsequence such that

$$(U_1+\mu)^{\frac{m-1}{2}}\nabla\mathcal{K}_s^\epsilon[U_1] \stackrel{\epsilon\to 0}{\longrightarrow} v \text{ in } L^2(B_R\times(0,T)) \text{ weakly }.$$

Moreover, it is trivial to show that $(U_1+\mu)^{-\frac{m-1}{2}} \stackrel{\epsilon\to 0}{\longrightarrow} (U_2+\mu)^{-\frac{m-1}{2}}$ in $L^2(B_R\times(0,T))$. Then

$$\nabla\mathcal{K}_s^\epsilon[U_1] = \frac{(U_1+\mu)^{\frac{m-1}{2}}}{(U_1+\mu)^{\frac{m-1}{2}}}\nabla\mathcal{K}_s^\epsilon[U_1] \stackrel{\epsilon\to 0}{\longrightarrow} \frac{v}{(U_2+\mu)^{\frac{m-1}{2}}} \text{ in } L^1(B_R\times(0,T)).$$

In particular we get that there exists a limit of $\nabla\mathcal{K}_s^\epsilon[U_1]$ as $\epsilon \to 0$ in any $L^p(B_R\times(0,T))$ with $1 \leq p \leq \infty$. Now we need to identify this limit. The following Lemma shows that $\nabla\mathcal{K}_s^\epsilon[U_1] \stackrel{\epsilon\to 0}{\longrightarrow} \nabla\mathcal{K}_s[U_2]$ in distributions, and so we can conclude convergence in $L^2(B_R\times(0,T))$.

Lemma 2.6.1. *Let $s \in (0,1)$ ($0 < s < 1/2$ if $N = 1$). Then*

(1) $\mathcal{K}_s^\epsilon[U_1] \xrightarrow{\epsilon\to 0} \mathcal{K}_s[U_2]$ *in* $L^1(B_R\times(0,T))$.

(2) $\displaystyle\int_0^T\int_{B_R}\mathcal{K}_s^\epsilon[U_1]\nabla\psi\,dxdt \xrightarrow{\epsilon\to 0} \int_0^T\int_{B_R}\mathcal{K}_s[U_2]\nabla\psi\,dxdt$ *for every* $\psi\in C_c^\infty(Q_T)$.

Proof. For the first part of the Lemma, we split the integral as follows,

$$\int_0^T\int_{B_R}(\mathcal{K}_s^\epsilon[U_1]-\mathcal{K}_s[U_2])dxdt = \int_0^T\int_{B_R}(\mathcal{K}_s^\epsilon[U_1]-\mathcal{K}_s[U_1])dxdt+\int_0^T\int_{B_R}(\mathcal{K}_s[U_1]-\mathcal{K}_s[U_2])dxdt.$$

Note that $\mathcal{K}_s[U_1]=K_s*U_1$ with $K_s\in L^1_{loc}(\mathbb{R}^N)$ and $\mathcal{K}_s^\epsilon[U_1]=K_s^\epsilon*U_1$ with $K_s^\epsilon=\rho_\epsilon*K_s$ where ρ_ϵ is a standard mollifier. Then the first integral on the right hand side goes to zero as $\epsilon\to 0$. The second integral goes to zero with ϵ as consequence of (2.6.4).

The second part of the Lemma is just a corollary of the first part.

$$\left|\int_0^T\int_{B_R}(\mathcal{K}_s^\epsilon[U_1]-\mathcal{K}_s[U_2])\nabla\psi dxdt\right| \le ||\nabla\psi||_\infty||\mathcal{K}_s^\epsilon[U_1]-\mathcal{K}_s[U_2]||_{L^1(B_R\times(0,T))}.$$

□

The remaining case $N=1$, $s\in(1/2,1)$ will be explained in Section 2.6.5. We conclude that,

$$\int_0^T\int_{B_R}d_\mu(U_1)\nabla\mathcal{K}_s^\epsilon[U_1]\nabla\phi dxdt \to \int_0^T\int_{B_R}d_\mu(U_2)\nabla\mathcal{K}_s[U_2]\nabla\phi dxdt,\quad \text{as } \epsilon\to 0.$$

Note that we can obtain also that $\nabla\mathcal{H}_s^\epsilon[U_1] \xrightarrow{\epsilon\to 0} \nabla\mathcal{H}_s[U_2]$ in $L^2(B_R\times(0,T))$ using the same argument. This allows us to pass to the limit in the energy estimates.

The conclusion of this step is that we have obtained a weak solution of the initial value problem ($P_{\delta\mu R}$) posed in $B_R\times[0,T]$ with homogeneous Dirichlet boundary conditions. The regularity of U_2, $\mathcal{H}_s[U_2]$ and $\mathcal{K}_s[U_2]$ is as stated before. We also have the energy formulas

$$\int_{B_R}F_\mu(U_2(t))dx+\delta\int_0^t\int_{B_R}\frac{|\nabla U_2|^2}{d_\mu(U_2)}dxdt+\int_0^t\int_{B_R}|\nabla\mathcal{H}_s[U_2]|^2dxdt=\int_{B_R}F_\mu(u_0)dx. \tag{2.6.5}$$

$$\frac{1}{2}\int_{B_R}|\mathcal{H}_s[U_2(t)]|^2dx+\delta\int_0^t\int_{B_R}|\nabla\mathcal{H}_s[U_2]|^2dx\,dt+\int_0^t\int_{B_R}(U_2+\mu)^{m-1}|\nabla\mathcal{K}_s[U_2]|^2dx\,dt \le \frac{1}{2}\int_{B_R}|\mathcal{H}_s[\widehat{u}_0]|^2dx.$$

The next step in the approximation process will not be passing to the limit as $\delta \to 0$. This limit would make us lose H^1 estimates for U_2 provided by the first energy estimate. Therefore, we keep the term $\delta\Delta U_2$ and deal first with the problem caused by the homogenous Dirichlet data on the boundary ∂B_R.

2.6.2 Limit as $R \to \infty$

We will now pass to the limit as $R \to \infty$. The estimates used in the limit on ϵ in Section 2.4.2 are also independent on R. Then the same technique may be applied here in order to pass to the limit as $R \to \infty$. Indeed, we get that $U_3 = \lim_{R\to\infty} U_2$ in $L^2(\mathbb{R}^N \times (0,T))$ is a weak solution of the problem in the whole space

$$(U_3)_t = \delta\Delta U_3 + \nabla \cdot \big((U_3+\mu)^{m-1}\nabla\mathcal{K}_s[U_3]\big) \quad x \in \mathbb{R}^N, \quad t > 0. \tag{$P_{\mu\delta}$}$$

This problem satisfies the property of conservation of mass, that we prove next.

Lemma 2.6.2. *Let $u_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Then the constructed non-negative solution of Problem ($P_{\mu\delta}$) satisfies*

$$\int_{\mathbb{R}^N} U_3(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx \text{ for all } t > 0. \tag{2.6.6}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ a cutoff test function supported in the ball B_{2R} and such that $\varphi \equiv 1$ for $|x| \le R$, we recall the construction in the Appendix 2.9.2. We get

$$\int_{B_{2R}} (U_3)_t\varphi dx = \delta\int_{B_{2R}} U_3\Delta\varphi dx - \int_{B_{2R}} (U_3+\mu)^{m-1}\nabla\mathcal{K}_s[U_3]\cdot\nabla\varphi dx = I_1 + I_2.$$

Since $U_3(t) \in L^1(\mathbb{R}^N)$ for any $t \ge 0$, we estimate the first integral as $I_1 = O(R^{-2})$ and then $I_1 \to 0$ as $R \to \infty$. For the second integral we have

$$I_2 = \int_{B_{2R}} \mathcal{K}_s[U_3]\nabla\cdot\big((U_3+\mu)^{m-1}\nabla\varphi\big)\,dx,$$

$$I_2 = (m-1)\int_{B_{2R}} \mathcal{K}_s[U_3](U_3+\mu)^{m-2}\nabla u\cdot\nabla\varphi dx + \int_{B_{2R}} \mathcal{K}_s[U_3](U_3+\mu)^{m-1}\Delta\varphi dx = I_{21}+I_{22}.$$

Since $\nabla U_3 \in L^2(\mathbb{R}^N)$ and $U_3 \in L^\infty(\mathbb{R}^N)$,

$$|I_{21}| \le C||(U_3+\mu)^{m-2}||_\infty\left(\int_{B_{2R}}|\nabla U_3|^2dx\right)^{1/2}\left(\int_{B_{2R}}|\mathcal{K}_s[U_3]|^2|\nabla\varphi|^2dx\right)^{1/2}.$$

Now $\nabla\varphi = O(R^{-1})$, $\nabla\varphi \in L^p$ with $p > N$, so we need $\mathcal{K}_s[U_3] \in L^q$ for $q < 2\dfrac{1}{1-\frac{1}{N/2}} = \dfrac{2N}{N-2}$ which is true since $\mathcal{K}[U_3] \in L^q$ for $q > q_0 = N/(N-2s)$, and $q_0 < 2N/(N-2)$ if $4s < N+2$. So, since $p > N$,

$$\begin{aligned} |I_{21}| &\le C\left(\int_{B_{2R}} |\nabla \mathcal{K}_s[U_3]|^q dx\right)^{1/q}\left(\int_{B_{2R}} |\nabla\varphi|^p dx\right)^{1/p} \\ &\le C\left(\int_{B_{2R}} R^{-p} dx\right)^{1/p} \le CR^{\frac{N-p}{p}} \overset{R\to\infty}{\longrightarrow} 0. \end{aligned}$$

For I_{22}, we will use the same trick of the previous section,

$$I_{22} = \int_{B_{2R}} \mathcal{K}_s[U_3]\left[(U_3+\mu)^{m-1} - \mu^{m-1}\right]\Delta\varphi dx + \mu^{m-1}\int_{B_{2R}} \mathcal{K}_s[U_3]\Delta\varphi dx = I_{221} + I_{222}.$$

Now,

$$I_{222} = \mu^{m-1}\int_{B_{2R}} U_3\mathcal{K}_s[\Delta\varphi]dx = \mu^{m-1}||U_3||_1 O(R^{-2+2s}) \overset{R\to\infty}{\longrightarrow} 0,$$

where we use the fact that $\mathcal{K}\Delta$ has homogeneity $2-2s > 0$ as a differential operator. Also,

$$I_{221} = \int_{B_{2R}} f'(\xi)U_3\mathcal{K}_s(U_3)\Delta\varphi dx,$$

where $f(s) = s^{m-1}$ and $\xi \in [\mu, \mu + U_3(x)]$. Again, since $U_3 \in L^\infty$, there exists a global bound for $f'(\xi)$, that is, $f'(\xi) \le (m-1)\max\{\mu^{m-2}, (\mu + ||U_3||_\infty)^{m-2}\}$ and so integral $I_{221} \to 0$ as $R \to \infty$ (details could be found in [25]).

In the limit $R \to \infty$, $\varphi \equiv 1$ and we get (2.6.6). $\square$

Consequence. The estimates done in Section 2.4.2 can be improved passing to the limit $R \to \infty$, since the conservation of mass (2.6.6) eliminates some of the integrals that presented difficulties when trying to obtain upper bounds independent of μ. Therefore, we compute the following terms in the energy estimate (2.6.5).

For $m \ne 2, 3$ we have

$$\begin{aligned} &\int_{B_R} F_\mu(u_0)dx - \int_{B_R} F_\mu(U_2)dx = \\ &= C\int_{B_R}[(u_0+\mu)^{3-m} - \mu^{3-m}]dx - \frac{1}{2-m}\mu^{2-m}\int_{B_R} u_0 dx \\ &- C\int_{B_R}[(U_2+\mu)^{3-m} - \mu^{3-m}]dx + \frac{1}{2-m}\mu^{2-m}\int_{B_R} U_2 dx \\ &\longrightarrow \quad C\int_{\mathbb{R}^N}[(u_0+\mu)^{3-m} - \mu^{3-m}]dx - C\int_{\mathbb{R}^N}[(U_3+\mu)^{3-m} - \mu^{3-m}]dx, \end{aligned} \tag{2.6.7}$$

as $R \to \infty$. We use the notation $C = \frac{1}{(2-m)(3-m)}$.

For $m=3$ we have

$$\begin{aligned}
&\int_{B_R} F_\mu(u_0)dx - \int_{B_R} F_\mu(U_2)dx = \\
&= -\int_{B_R} \log\left(1+\frac{u_0}{\mu}\right)dx + \frac{1}{\mu}\int_{B_R} u_0 dx + \int_{B_R} \log\left(1+\frac{U_2}{\mu}\right)dx - \frac{1}{\mu}\int_{B_R} U_2 dx \\
&\longrightarrow \quad \int_{\mathbb{R}^N} \log\left(1+\frac{U_3}{\mu}\right)dx - \int_{\mathbb{R}^N} \log\left(1+\frac{u_0}{\mu}\right)dx \quad \text{as } R\to\infty. \qquad (2.6.8)
\end{aligned}$$

The following theorem summarizes the results obtained until now.

Theorem 2.6.3. *Let $m>1$ and $u_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ be non-negative. Then there exists a weak solution U_3 of Problem $(P_{\mu\delta})$ posed in $\mathbb{R}^N\times(0,T)$ with initial data u_0. Moreover, $U_3 \in L^\infty([0,\infty) : L^1(\mathbb{R}^N))$, and for all $t>0$ we have*

$$\int_{\mathbb{R}^N} U_3(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$$

and $||U_3(\cdot,t)||_\infty \le ||u_0||_\infty$. The following energy estimates also hold:

(i) First energy estimate:

- *If $m=3$,*

$$\begin{aligned}
\delta\int_0^t\int_{\mathbb{R}^N} \frac{|\nabla U_3|^2}{(U_3+\mu)^2}dxdt \quad &+ \quad \int_0^t\int_{\mathbb{R}^N} |\nabla \mathcal{H}_s[U_3]|^2 dxdt + \int_{\mathbb{R}^N}\log\left(\frac{u_0}{\mu}+1\right)dx \qquad (2.6.9)\\
&\le \quad \int_{\mathbb{R}^N}\log\left(\frac{U_3(t)}{\mu}+1\right)dx.
\end{aligned}$$

- *If $m\neq 2,3$ and*

$$\begin{aligned}
\delta\int_0^t\int_{\mathbb{R}^N} \frac{|\nabla U_3|^2}{(U_3+\mu)^{m-1}}dxdt \quad &+ \quad \int_0^t\int_{\mathbb{R}^N} |\nabla \mathcal{H}_s[U_3]|^2 dxdt + \qquad (2.6.10)\\
+C\int_{\mathbb{R}^N}\left[(U_3(t)+\mu)^{3-m}-\mu^{3-m}\right]dx \quad &\le \quad C\int_{\mathbb{R}^N}\left[(u_0+\mu)^{3-m}-\mu^{3-m}\right]dx
\end{aligned}$$

where $C=C(m)=\frac{1}{(2-m)(3-m)}$.

(ii) Second energy estimate:

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N}|\mathcal{H}_s[U_3(T)]|^2dx + \delta\int_0^T\int_{\mathbb{R}^N}|\nabla\mathcal{H}_s[U_3]|^2dx\,dt + \int_0^T\int_{\mathbb{R}^N}(U_3+\mu)^{m-1}|\nabla\mathcal{K}_s[U_3]|^2dx\,dt\\
\le \frac{1}{2}\int_{\mathbb{R}^N}|\mathcal{H}_s[u_0]|^2dx.
\end{aligned}$$

2.6.3 Limit as $\mu \to 0$

Similarly to the previous limits we can prove that $U_4 = \lim_{\mu \to 0} U_3$ in $L^2(\mathbb{R}^N \times (0,T))$ when $m \in (1,3)$. Then U_4 will be a solution of problem

$$(U_4)_t = \delta \Delta U_4 + \nabla \cdot \left(U_4^{m-1} \nabla \mathcal{K}_s[U_4] \right) \quad x \in \mathbb{R}^N, \quad t > 0. \tag{P_δ}$$

In order to pass to the limit, we need to find uniform bounds on $\mu > 0$ for terms 3 and 4 of the energy estimates (2.6.9) and (2.6.10).

Uniform upper bounds

• Case $m \in (1,2)$. By the Mean Value Theorem,

$$\frac{1}{(m-2)(3-m)} \int_{\mathbb{R}^N} \left[(u_0+\mu)^{3-m} - \mu^{3-m} \right] dx \le \frac{1}{(m-2)} \int_{\mathbb{R}^N} (u_0+\mu)^{2-m} u_0 dx$$
$$\le \frac{(||u_0||_\infty + 1)^{2-m}}{m-2} \int_{\mathbb{R}^N} u_0 dx.$$

This bound is independent of μ.

• Case $m \in (2,3)$. The function $f(\zeta) = \zeta^{3-m}$ is concave and so $f(U_3+\mu) \le f(\mu) + f(U_3)$. In this way,

$$\frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} \left[(U_3(t)+\mu)^{3-m} - \mu^{3-m} \right] dx \le \frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} U_3(t)^{3-m} dx.$$

The last integral is finite due to the exponential decay for U_3 that we proved in Section 2.5. In this way, the last estimate is uniform in μ.

The limit is a solution of the new problem (P_δ). The argument from Section 2.6.1 does not apply for the limit

$$\int_0^T \int_{\mathbb{R}^N} (U_3+\mu)^{m-1} \nabla \mathcal{K}_s[U_3] \nabla \phi dx dt \stackrel{\mu \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla \mathcal{K}_s[U_4] \nabla \phi dx dt. \tag{2.6.11}$$

In order to show that this convergence holds, we note that from the first energy estimate we get that

$$\nabla \mathcal{H}_s[U_3] \in L^2((0,T) : L^2(\mathbb{R}^N))$$

uniformly on μ. Then $\nabla \mathcal{K}_s[U_3] = \mathcal{H}_s[\nabla \mathcal{H}_s[U_3]] \in L^2((0,T) : H^s(\mathbb{R}^N))$. Since for any bounded domain Ω, $H^s(\Omega)$ is compactly embedded in $L^2(\Omega)$ then $\nabla \mathcal{K}_s[U_3] \to \nabla \mathcal{K}_s[U_4]$ as $\mu \to 0$ in $L^2(\Omega)$. Then we have the convergence (2.6.11) since $U_3 \in L^\infty(\mathbb{R}^N)$ and ϕ is compactly supported.

Remarks. • In the case $m = 2$ the corresponding term is $\int_{\mathbb{R}^N} U_3 \log^-(U_3 + \mu)dx$ which is uniformly bounded if U_3 has an exponential tail. This has been proved by Caffarelli and Vázquez in [25]. We do not repeat the proof here.

• The case $m \geq 3$ is more difficult since we can not find uniform estimates in $\mu > 0$ for the energy estimates that allow us to pass to the limit.

2.6.4 Limit as $\delta \to 0$

We will prove that there exists a limit $u = \lim_{\delta \to 0} U_4$ in $L^2(\mathbb{R}^N \times (0,T))$ and that $u(x,t)$ is a weak solution to Problem (2.1.1). Thus, we conclude the proof of Theorem 2.1.2 stated in the introduction of this chapter.

We comment on the differences that appear in this case. From the first energy estimate we have that

$$\delta \int_0^T \int_{\mathbb{R}^N} \frac{|\nabla U_4|^2}{U_4^{m-1}} dxdt \leq C(m, u_0),$$

which gives us that $\delta \nabla U_4 \in L^2(Q_T)$ since $U_4 \in L^\infty(Q_T)$. Then, as in Section 2.6.1, we have that $\delta \Delta U_4 \in H^{-1}(\mathbb{R}^N)$ uniformly in δ. Also $\nabla(U_4^{m-1} \nabla \mathcal{K}_s[U_4]) \in H^{-1}(\mathbb{R}^N)$ as before. Then $(U_4)_t \in H^{-1}(\mathbb{R}^N)$ independently on δ. Therefore we use the compactness argument of Simon to obtain that there exists a limit

$$U_4(x,t) \to u(x,t) \quad L^2((0,T) \times \mathbb{R}^N).$$

Now we show that u is the weak solution of Problem (2.1.1). It is trivial that $\delta \int_0^T \int_{\mathbb{R}^N} U_4 \Delta \phi \to 0$ as $\delta \to 0$. On the other hand, $\nabla \mathcal{K}_s[U_4] = \mathcal{H}_s[\nabla \mathcal{H}_s[U_4]] \in L^2_{loc}(Q_T)$ uniformly on $\delta > 0$ since $\nabla \mathcal{H}_s[U_4] \in L^2(Q_T)$ uniformly on $\delta > 0$. In this way, $\nabla \mathcal{K}_s[U_4]$ has a weak limit in $L^2_{loc}(Q_T)$. As in Lemma 2.6.1 (2) we can identify this limit and so on, $\nabla \mathcal{K}_s[U_4] \to \nabla \mathcal{K}_s[u]$ weakly in $L^2_{loc}(Q_T)$ as $\delta \to 0$ and therefore

$$\int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla \mathcal{K}_s[U_4] \nabla \phi dxdt \overset{\delta \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla \mathcal{K}_s[u] \nabla \phi dxdt.$$

since $U_4^{m-1} \to u^{m-1}$ in $L^2_{loc}(Q_T)$ as $\delta \to 0$.

2.6.5 Dealing with the case $N = 1$ and $1/2 < s < 1$

As we have commented before, the operator $\mathcal{K}_s$ is not well defined when $N = 1$ and $1/2 < s < 1$ since the kernel $|x|^{1-2s}$ does not decay at infinity, indeed it grows. It makes no sense to think of equation (2.6.12) in terms of a pressure as before. This is maybe not very convenient, but it is not an essential problem, since equation (2.1.1) can be considered in the following sense:

$$u_t(t,x) = \nabla \cdot \big(u^{m-1}(\nabla \mathcal{K}_s)[u]\big) \text{ for } x \in \mathbb{R}^N,\ t>0, \tag{2.6.12}$$

where the combined operator $(\nabla \mathcal{K}_s)$ is defined as the convolution operator

$$(\nabla \mathcal{K}_s)[u] := (\nabla K_s) * u \quad \text{ with } \quad K_s(x) = \frac{c_s}{|x|^{1-2s}}.$$

Other authors that dealt with $N=1$ have considered operator $(\nabla \mathcal{K}_s)$ before. They use the notation ∇^{2s-1} to refer to it. Note that

$$\nabla K_s(x) = (-1+2s)c_s \frac{x}{|x|^{3-2s}},$$

and so, $\nabla K_s \in L^1_{loc}(\mathbb{R})$ for $N=1$ and $1/2 < s < 1$. Moreover, $(\nabla \mathcal{K}_s)$ is an integral operator in this range. As in Subsection 2.2.2, the operator $(\nabla \mathcal{K}_s)$ is approximated by $(\nabla \mathcal{K}_s)^\epsilon$ defined as

$$(\nabla \mathcal{K}_s)^\epsilon[u] = (\nabla K_s)^\epsilon * u \quad \text{ where } \quad (\nabla K_s)^\epsilon = \rho_\epsilon * (\nabla K_s).$$

Note that

$$(\nabla K_s)^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \nabla K_s \quad \text{in} \quad L^1_{loc}(\mathbb{R}), \tag{2.6.13}$$

since $\nabla K_s \in L^1_{loc}(\mathbb{R})$. It is still true that

$$(\nabla \mathcal{K}_s)[u] = \mathcal{H}_s\left[\nabla \mathcal{H}_s[u]\right],$$

since the operator $\mathcal{H}_s$ is well defined for any $s \in (0,1)$ even in dimension $N=1$.

In this way, almost all the arguments from Section 2.6 apply by replacing $\nabla(\mathcal{K}_s(u))$ for $(\nabla \mathcal{K}_s)(u)$. The only exception is Lemma 2.6.1 where the weak $L^2(\mathbb{R})$ limit of $\nabla K_s[U_1]$ is identified. This argument is replaced by the following Lemma:

Lemma 2.6.4. *Let N=1 and* $1/2 < s < 1$. *Then*

$$\int_0^T \int_{B_R} U_1 (\nabla \mathcal{K}_s)^\epsilon[\psi]\, dxdt \overset{\epsilon\to 0}{\longrightarrow} \int_0^T \int_{B_R} U_2 (\nabla \mathcal{K}_s)[\psi]\, dxdt \qquad \forall \psi \in C_c^\infty(Q_T).$$

Proof.

$$\begin{aligned}\int_0^T \int_{B_R} U_1 (\nabla \mathcal{K}_s)^\epsilon[\psi] - U_2 (\nabla \mathcal{K}_s)[\psi]\, dxdt &= \int_0^T \int_{B_R} (U_1 - U_2)(\nabla \mathcal{K}_s)^\epsilon[\psi]\, dxdt \\ &\quad + \int_0^T \int_{B_R} U_2\big((\nabla \mathcal{K}_s)^\epsilon[\psi] - (\nabla \mathcal{K}_s)[\psi]\big)\, dxdt.\end{aligned}$$

The first integral on the right hand side goes to zero with ϵ since $||(\nabla\mathcal{K}_s)^\epsilon[\psi]||_{L^\infty(\mathbb{R})} \le K$ for some positive constant K which does not depend on ϵ and $U_1 \to U_2$ as $\epsilon \to 0$ in $L^2(B_R \times (0,T))$. The second integral also goes to zero as consequence of (2.6.13) and the fact that $U_2 \in L^\infty(\mathbb{R})$ uniformly on ϵ. □

2.7 Finite propagation property for $m \in [2, 3)$

In this section we will prove that compactly supported initial data $u_0(x)$ determine the solutions $u(x,t)$ that have the same property for all positive times.

Theorem 2.7.1. *Let $m \ge 2$. Assume u is a bounded solution, $0 \le u \le L$, of Equation (2.1.1) with $\mathcal{K} = (-\Delta)^{-s}$ with $0 < s < 1$ ($0 < s < 1/2$ if $N = 1$), as constructed in Theorem 2.1.3. Assume that u_0 has compact support. Then $u(\cdot,t)$ is compactly supported for all $t > 0$. More precisely, if $0 < s < 1/2$ and u_0 is below the "parabola-like" function*

$$U_0(x) = a(|x| - b)^2,$$

for some $a, b > 0$, with support in the ball $B_b(0)$, then there is a constant C large enough, such that

$$u(x,t) \le a(Ct - (|x| - b))^2.$$

Actually, we can take $C(L,a) = C(1,1)L^{m-\frac{3}{2}+s}a^{\frac{1}{2}-s}$. For $1/2 \le s < 1$ a similar conclusion is true, but $C = C(t)$ is an increasing function of t and we do not obtain a scaling dependence of L and a.

Proof. The method is similar to the tail control section. We assume $u(x,t) \ge 0$ has bounded initial data $u_0(x) = u(x,t_0) \le L$, and also that u_0 is below the parabola $U_0(x) = a(|x| - b)^2$, $a, b > 0$. Moreover the support of U_0 is the ball of radius b and the graphs of u_0 and U_0 are strictly separated in that ball. We take as comparison function $U(x,t) = a(Ct - (|x| - b))^2$ and argue at the first point in space and time where $u(x,t)$ touches U from below. The fact that such a first contact point happens for $t > 0$ and $x \ne \infty$ is justified by regularization, as before. We put $r = |x|$.

By scaling we may put $a = L = 1$. We denote by (x_c, t_c) this contact point where we have $u(x_c,t_c) = U(x_c,t_c) = (b + Ct_c - |x_c|)^2$. The contact can not be at the vanishing point $|x_f(t_c)| := b + Ct_c$ of the barrier and this will be proved in Lemma 2.7.2. We consider that x_c lies at a distance $h > 0$ from $|x_f(t_c)| = b + Ct_c$ (the boundary of the support of the parabola $U(x,t)$ at time t_c), that is

$$b + Ct_c - |x_c| = h > 0.$$

Note that since $u \le 1$ we must have $|h| \le 1$. Assuming that u is also C^2 smooth, since we deal with a first contact point (x_c, t_c), we have that $u = U$, $\nabla(u-U) = 0$, $\Delta(u-U) \le 0$, $(u-U)_t \ge 0$, that is

$$u(x_c, t_c) = h^2, \quad u_r = -2h, \quad \Delta u \le 2N, \quad u_t \ge 2Ch.$$

For $p = \mathcal{K}_s(u)$ and using the equation $u_t = (m-1)u^{m-2}\nabla u \cdot \nabla p + u^{m-1}\Delta p$, we get the inequality

$$2Ch \le 2(m-1)h^{2m-3}\left(-\overline{p_r} + \frac{h}{2}\overline{\Delta p}\right), \tag{2.7.1}$$

where $\overline{p_r}$ and $\overline{\Delta p}$ are the values of p_r and Δp at the point (x_c, t_c). In order to get a contradiction, we will use estimates for the values of $\overline{p_r}$ and $\overline{\Delta p}$ already proved in [25] (see Theorem 5.1. of [25])

$$-\overline{p_r} \le K_1 + K_2 h^{1+2s} + K_3 h, \quad \overline{\Delta p} \le K_4. \tag{2.7.2}$$

Therefore, inequality (2.7.1) combined with the estimates (2.7.2) implies that

$$2C \le 2(m-1)h^{2m-4}\left(K_1 + K_2 h^{1+2s} + Kh\right), \tag{2.7.3}$$

which is impossible for C large (independent of h), since $m > 2$ and $|h| \le 1$. Therefore, there cannot be a contact point with $h \ne 0$. In this way we get a minimal constant $C = C(N, s)$ for which such contact does not take place.

Remark: For $m < 2$, we do not obtain a contradiction in the estimate (2.7.3), since the term $K_1 h^{2m-4}$ can be very large for small values of $|h|$.

• **Reduction. Dependence on L and a.** The equation is invariant under the scaling

$$\widehat{u}(x,t) = Au(Bx, Tt) \tag{2.7.4}$$

with parameters $A, B, T > 0$ such that $T = A^{m-1}B^{2-2s}$.

Step I. We prove that if u has height $0 \le u(x,t) \le 1$ and initially satisfies $u(x,0) = u_0(x) \le (|x|-b)^2$ then $u(x,t) \le U(x,t) = (Ct - (|x|-b))^2$ for all $t > 0$.

Step II. We search for parameters A, B, T for which the function $\widehat{u}$ is defined by (2.7.4) satisfies

$$0 \le \widehat{u}(x,t) \le L, \quad \widehat{u}(x,0) \le \widehat{a}(|x| - \widehat{b})^2.$$

An easy computation gives us

$$A = L, \quad AB^2 = \widehat{a}, \quad \widehat{b} = b/B.$$

Moreover, by the relation between A, B and T we obtain $A = L$, $B = (\widehat{a}/L)^{1/2}$ and then $T = L^{m-2+s}\widehat{a}^{1-s}$. Then $\widehat{u}(x,t)$ is below the upper barrier $\widehat{U}(x,t) = \widehat{a}(\widehat{C}t - (|x| - \widehat{b}))^2$ where the new speed is given by

$$\widehat{C} = CA^{m-1}B^{1-2s} = CL^{m-\frac{3}{2}+s}\,\widehat{a}^{\frac{1}{2}-s}.$$

• **Case** $1/2 \le s < 1$**.** The proof relies on estimating the term $\partial_r p$ at a possible contact point. This is independent on m and it was done in [25]. □

Lemma 2.7.2. *Under the assumptions of Theorem 2.7.1 there is no contact between* $u(x,t)$ *and the parabola* $U(x,t)$*, in the sense that strict separation of* u *and* U *holds for all* $t > 0$ *if* C *is large enough.*

Proof. We want to eliminate the possible contact of the supports at the lower part of the parabola, that is the minimum $|x| = Ct + b$. Instead of analyzing the possible contact point, we proceed by a change in the test function that we replace by

$$U_\epsilon(x,t) = \begin{cases} (Ct - (|x| - b))^2 + \epsilon(1 + Dt) & \text{for } |x| \le b + Ct, \\ \epsilon(1 + Dt), & \text{for } |x| \ge b + Ct. \end{cases}$$

The function U_ϵ is constructed from the parabola U by a vertical translation $\epsilon(1 + Dt)$ and a lower truncation with $1 + Dt$ outside the ball $\{|x| \le b + Ct\}$. Here $0 < \epsilon < 1$ is a small constant and $D > 0$ will be suitable chosen.

We assume that the solution $u(x,t)$ starts as $u(x,0) = u_0(x)$ and touches for the first time the parabola U_ϵ at $t = t_c$ and spatial coordinate x_c. The contact point can not be a ball $\{|x| \le b + Ct\}$ since U_ϵ is a parabola here and this case was eliminated in the previous Theorem 2.7.1. Consider now the case when the first contact point between $u(x,t)$ and $U_\epsilon(x,t)$ is when $|x_c| \ge b + Ct_c$. At the contact point we have that $u = U_\epsilon$, $\nabla(u - U_\epsilon) = 0$, $\Delta(u - U_\epsilon) \le 0$, $(u - U_\epsilon)_t \ge 0$. In this region the spatial derivatives of U_ϵ are zero, hence the equation gives us

$$D\epsilon = (\epsilon(1 + Dt_c))^{m-1}\overline{\Delta p},$$

where $\overline{\Delta p}$ is the value of $\Delta p = (-\Delta)^{1-s}u$ at the point (x_c, t_c). Since ϵ is small we get that the bound $u(x,t) \le U_1(x,t)$ is true for all $|x| \le \mathbb{R}^N$. This allows us to prove that that $\overline{\Delta p}$ is bounded by a constant K. We obtain that $D\epsilon \le (\epsilon(1 + Dt_c))^{m-1}K$. Since $m \ge 2$ and $\epsilon < 1$, this implies that

$$D \le (1 + Dt_c)^{m-1}K.$$

We obtain a contradiction for large D, for example $D = 2K$, and for

$$t_c < T_c = \frac{1}{2K}\left(2^{1/(m-1)} - 1\right).$$

Therefore, we proved that a contact point between u and U_ϵ is not posible for $t < T_c$, and thus $u(x,t) \le U_\epsilon(x,t)$ for $t < T_c$. The estimate on t_c is uniform in ϵ and we obtain in the limit $\epsilon \to 0$ that

$$u(x,t) \le U(x,t) = (Ct - (|x| - b)) \quad \text{for } t < \frac{1}{2K}\left(2^{1/(m-1)} - 1\right).$$

As a consequence, the support of $u(x,t)$ is bounded by the line $|x| = Ct + b$ in the time interval $[0, T_c)$. The comparison for all times can be proved with an iteration process in time.

• Regularity requirements. Using the smooth solutions of the approximate equations, the previous conclusions hold for any constructed weak solution.

□

Remark. The following result about the free boundary is valid only for $s < 1/2$ and for solutions with bounded and compactly supported initial data. The result is a direct consequence of the parabolic barrier study done in the previous section. Since that barrier does not depend explicitly on m if $m \ge 2$, the proof presented in [25] is valid here. By free boundary $\mathcal{FB}(u)$ we mean, the topological boundary of the support of the solution $S(u) = \overline{\{(x,t) : u(x,t) > 0\}}$.

Corollary 2.7.3 (Growth estimates of the support). *Let u_0 be bounded with $u_0(x) = 0$ for $|x| > R$ for some $R > 0$. If $(x,t) \in \mathcal{FB}(u)$ then $x \le R + Ct^{1/(2-2s)}$, where $C = C(\|u_0\|_\infty, N, s)$.*

2.7.1 Persistence of positivity

This property is also interesting in the sense that avoids the possibility of degeneracy points for the solutions. In particular, assuming that the solutions are continuous, it implies the non-shrinking of the support. Due to the nonlocal character of the operator, the following theorem can be proved only for a certain class of solutions.

Lemma 2.7.4. *Let u be a weak solution as constructed in Theorem 2.1.3 and assume that the initial data $u_0(x)$ is radially symmetric and non-increasing in $|x|$. Then $u(x,t)$ is also radially symmetric and non-increasing in $|x|$.*

Proof. The operators in the approximate problem ($P_{\epsilon\delta\mu R}$) are invariant under rotation in the space variable. Since the solution of problem ($P_{\epsilon\delta\mu R}$) is unique, then we obtain that $u(x,t)$ is radially symmetric. □

Theorem 2.7.5. *Let u be a weak solution as constructed in Theorem 2.1.3 and assume that it is a radial function of the space variable $u(|x|,t)$ and is non-increasing in $|x|$. If $u_0(x)$ is positive in a neighborhood of a point x_0, then $u(x_0,t)$ is positive for all times $t>0$.*

Proof. A similar technique as the one presented in the tail analysis is used for this proof, but with what we call true subsolutions. Assume $u_0(x)\geq c>0$ in a ball $B_R(x_0)$. By translation and scaling we can also assume $c=R=1$ and $x_0=0$. Again, we will study a possible first contact point with a barrier that shrinks quickly in time, like

$$U(x,t)=e^{-at}F(|x|), \tag{2.7.5}$$

with $F:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\geq 0}$ to be chosen later and $a>0$ large enough. Choose $F(0)=1/2$, $F(r)=0$ for $r\geq 1/2$ and $F'(r)\leq 0$ for all $r\in\mathbb{R}_{\geq 0}$. The contact point (x_c,t_c) is sought in $B_{1/2}(0)\times(0,\infty)$. By approximation we can assume that u is positive everywhere so there are no contact points at the parabolic border. At the possible contact point (x_c,t_c) we have

$$u(x_c,t_c)=U(x_c,t_c),\quad u_t(x_c,t_c)\leq U_t(x_c,t_c)=-aU(x_c,t_c),$$

$$\nabla u(x_c,t_c)=\nabla U(x_c,t_c)=e^{-at_c}F'(|x_c|)\mathbf{e_r},\quad \mathbf{e_r}=x_c/|x_c|.$$

We recall the equation

$$u_t=(m-1)u^{m-2}\nabla u\nabla p+u^{m-1}\Delta p.$$

Then at the contact point (x_c,t_c) we have

$$-aU=U_t\geq u_t=(m-1)U^{m-2}\nabla U\overline{\nabla p}+U^{m-1}\overline{\Delta p},$$

where $\overline{\Delta p}=\Delta p(x_c,t_c)$. Then

$$-ae^{-at_c}F(|x_c|)\geq(m-1)e^{-a(m-2)t_c}F(|x|)^{m-2}e^{-at_c}F'(|x_c|)\overline{p_r}+e^{-a(m-1)t_c}F(|x_c|)^{m-1}\overline{\Delta p}.$$

According to [25] we know that the term $F'(|x|)\,\overline{p_r}\geq 0$ and $\overline{\Delta p}$ is bounded uniformly. Therefore

$$-ae^{-at_c}F(|x_c|)\geq e^{-a(m-1)t_c}F(|x_c|)^{m-1}\overline{\Delta p}.$$

Simplifying and using that $m \geq 2$, $\overline{\Delta p}$ is bounded uniformly and also F is bounded, we obtain

$$a \leq -e^{-a(m-2)t_c} F(|x_c|)^{m-2} \overline{\Delta p} \leq K e^{-a(m-2)t_c} \leq K.$$

This is not true if $a > K$ and we arrive at a contradiction. $\square$

Remark. There exist counterexamples on the persistence of positivity property when the hypothesis of Theorem 2.7.5 are not satisfied. In [25] (Theorem 6.2) the authors construct an explicit counterexample by taking an initial data with not connected support.

2.8 Infinite propagation speed in the case $1 < m < 2$ and $N = 1$

In this section we will consider model (2.1.1)

$$\partial_t u = \partial_x \cdot (u^{m-1} \partial_x p), \quad p = (-\Delta)^{-s} u, \tag{2.8.1}$$

for $x \in \mathbb{R}$, $t > 0$ and $s \in (0,1)$. We take compactly supported initial data $u_0 \geq 0$ such that $u_0 \in L^1_{\text{loc}}(\mathbb{R})$. We want to prove infinite speed of propagation of the positivity set for this problem. This is not easy, hence we introduce the integrated solution v, given by

$$v(x,t) = \int_{-\infty}^{x} u(y,t)\, dy \geq 0 \quad \text{for } t > 0,\ x \in \mathbb{R}. \tag{2.8.2}$$

Therefore $v_x = u$ and $v(x,t)$ is a solution of the equation

$$\partial_t v = -|v_x|^{m-1} (-\Delta)^{\alpha} v, \tag{2.8.3}$$

in some sense that we will make precise. The exponents α and s are related by $\alpha = 1-s$. The technique of the integrated solution has been extensively used in the standard Laplacian case to relate the porous medium equation with its integrated version, which is the p-Laplacian equation, always in 1D, with interesting results, see e. g. [60]. The use of this tool in [13] for fractional Laplacians in the case $m = 2$ was novel and very fruitful. We consider equation (2.8.3) with initial data

$$v(x,0) = v_0(x) := \int_{-\infty}^{x} u_0(x)\, dx \quad \text{for all } x \in \mathbb{R}. \tag{2.8.4}$$

Note that $v(x,t)$ is a non-decreasing function in the space variable x. Moreover, since $u(x,t)$ enjoys the property of conservation of mass, then $v(x,t)$ satisfies (see Figure 2.1)

$$\lim_{x\to-\infty} v(x,t) = 0, \quad \lim_{x\to+\infty} v(x,t) = M$$

for all $t \geq 0$. We devote a separate study to the solution v of the integrated problem (2.8.3) in Section 2.8.3. The validity of the maximum principle for equation (2.8.3) allows to prove a clean propagation theorem for v.

Theorem 2.8.1 (Infinite speed of propagation). *Let v be the solution of Problem (2.8.3)-(2.8.4), and assume that $u_0 \geq 0$ is compactly supported. Then $0 < v(x,t) < M$ for all $t > 0$ and $x \in \mathbb{R}$.*

The use of the integrated function is what forces us to work in one space dimension. The result continues the theory of the porous medium equation with potential pressure, by proving that model (2.8.1) has different propagation properties depending on the exponent m by the ranges $m \geq 2$ and $1 < m < 2$. Such a behavior is well known to be typical for the classical Porous Medium Equation $u_t = \Delta u^m$, recovered formally for $s = 0$, which has finite propagation for $m > 1$ and infinite propagation for $m \leq 1$. Therefore, our result is unexpected, since it shows that for the fractional diffusion model the separation between finite and infinite propagation is moved to $m = 2$.

Proof of Theorem 2.1.5, part b). This weaker result follows immediately. In fact, in Theorem 2.8.1 we prove that $v(x,t)$ defined by (2.8.2) is positive for every $t > 0$ if $x \in \mathbb{R}$. Therefore for every $t > 0$ there exist points x arbitrary far from the origin such that $u(x,t) > 0$.

If moreover, u_0 is radially symmetric and non-increasing in $|x|$ and u inherits the symmetry and monotonicity properties of the initial data as proved in Lemma 2.7.4. This ensures that u can not take zero values for any $x \in \mathbb{R}$ and $t > 0$.

□

2.8.1 Study of the integrated problem

• Connection between Model (2.8.1) and Model (2.8.3)

We explain how the properties of the Model (2.8.1) with $N = 1$ can be obtained via a study of the properties of the integrated equation (2.8.3). We consider equation (2.8.1) with compactly supported initial data u_0 such that $u_0 \geq 0$. Let us say that $\operatorname{supp} u_0 \subset [-R, R]$, where $R > 0$. Therefore, the corresponding initial data to be considered for the integrated problem is $v_0(x) = \int_{-\infty}^{x} u_0(y)dy$, for all $x \in \mathbb{R}$. Then $v_0 : \mathbb{R} \to [0, \infty)$ and has the properties

$$v_0(x) = 0 \text{ for } x < -R, \quad v_0(x) = M \text{ for } x > R, \quad v_0'(x) \geq 0 \text{ for } x \in (-R, R), \tag{2.8.5}$$

where $\mathbb{R} > 0$ is fixed from the beginning and $M = \int_{\mathbb{R}} u_0(x)dx$ is the total mass.

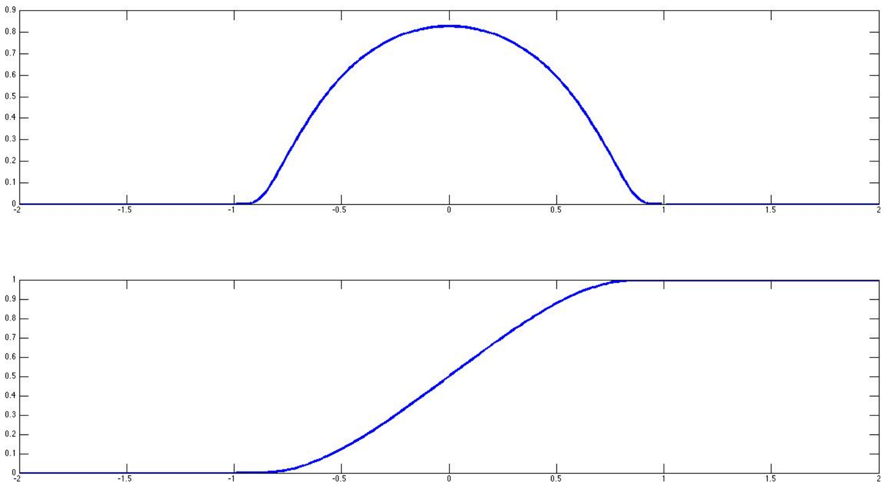

FIGURE 2.1: Typical compactly supported initial data for models (2.8.1) and (2.8.3).

2.8.2 Regularity

Proposition 2.8.2. *The solution $v : [0, T] \times \mathbb{R} \to [0, \infty)$ of Problem (2.8.3) defined by formula*
$v(x, t) = \displaystyle\int_{-\infty}^{x} u(y, t)dy$ *is continuous in space and time.*

Proof. **I. Preliminary estimates.** Since $v_x(x, t) = u(x, t)$, where u is the solution of Problem (2.1.1), then by the estimates of Section 2.6.1 we have the following:

• $v_x = u \in L^\infty([0, T] : L^\infty(\mathbb{R}))$, therefore $v \in L^\infty([0, T] : \mathrm{Lip}(\mathbb{R}))$, where $\mathrm{Lip}(\mathbb{R})$ is the space of Lipschitz continuous functions on $\mathbb{R}$. In particular, $v \in L^\infty([0, T] : \mathrm{Lip}(B_R))$ for every $B_R \subset \mathbb{R}$.

• We have $(v_t)_x = u_t = \partial_x(u^{m-1}\partial_x(-\Delta)^{-s}u)$ in the sense of distributions. Then $v_t \in L^2([0, T] : L^2(B))$ for every set $B \subset \mathbb{R}$, with $|B| < +\infty$. The proof is as follows. The first equality holds in the distributions sense, that is

$$\int_0^T \int_{\mathbb{R}} v_t \varphi_x \, dx \, dt := -\int_0^T \int_{\mathbb{R}} v(\varphi_x)_t \, dx \, dt = \int_0^T \int_{\mathbb{R}} u^{m-1}\partial_x(-\Delta)^{-s}u \, \varphi_x \, dx \, dt, \quad \forall \varphi \in C_0^\infty(\mathbb{R}\times[0, T]).$$

This implies that $v_t = u^{m-1}\partial_x(-\Delta)^{-s}u$ a.e. in $\mathbb{R}$. Then, using the second energy estimate (2.3.3), we obtain

$$\begin{aligned}\|v_t\|^2_{L^2([0,T]:L^2(B))} &= \|u^{m-1}\partial_x(-\Delta)^{-s}u\|^2_{L^2([0,T]:L^2(B))}\\ &\le \|u\|^{m-1}_{L^\infty(\mathbb{R})}\int_0^T\int_B u^{m-1}|\partial_x(-\Delta)^{-s}u|^2dxdt < +\infty.\end{aligned}$$

II. Continuity in time. Let $(x_0,t_0)\in\mathbb{R}\times[0,T]$. Let $(x,t_1)\in\mathbb{R}\times[0,T]$ and $h:=x-x_0$. Let $B=[x_0,x_1]$. Then

$$|v(x_0,t_1)-v(x_0,t_0)| \le |v(x_0,t_1)-v(x,t_1)| + |v(x,t_0)-v(x_0,t_0)| + |v(x,t_1)-v(x,t_0)|.$$

We know $v\in \mathrm{Lip}_x(\mathbb{R})$; let L the corresponding Lipschitz constant. Then

$$\begin{aligned}|v(x_0,t_1)-v(x_0,t_0)| &\le 2Lh + \frac{1}{h}\int_{x_0}^x |v(y,t_1)-v(y,t_0)|dy\\ &\le 2Lh + \frac{1}{h}\int_{x_0}^x \left|\int_{t_0}^{t_1} v_t dt\right| dy \le 2Lh^2 + \int_{x_0}^x\int_{t_0}^{t_1} |v_t|\, dydt\\ &\le 2Lh + \frac{1}{h}|B|^{1/2}|t_1-t_0|^{1/2}\|v_t\|^2_{L^2([0,T]:L^2(B)}\\ &= 2Lh + \frac{|t_1-t_0|^{1/2}}{h^{1/2}}\|v_t\|^2_{L^2([0,T]:L^2(B)}.\end{aligned}$$

Optimizing, we choose $h \sim \frac{|t_1-t_0|^{1/2}}{h^{1/2}}$, that is $h\sim(t_1-t_0)^{3/2}$, and we obtain that

$$|v(x_0,t_1)-v(x_0,t_0)| \le K|t_1-t_0|^{1/3}.$$

This estimate holds uniformly in $x\in\mathbb{R}$ and it proves that $v(x,t)$ is Holder continuous in time. In particular $v\in C([0,T]:C(\mathbb{R}))$. $\square$

2.8.3 Viscosity solutions

Notion of solution. We define the notions of viscosity sub-solution, super-solution and solution in the sense of Crandall-Lions [36]. The definition will be adapted to our problem by considering the time dependency and also the nonlocal character of the Fractional Laplacian operator. For a presentation of the theory of viscosity solutions to more general integro-differential equations we refer to Barles and Imbert [9].

It will be useful to make the notations:

$\mathrm{USC}(Q) = \{$upper semi-continuous functions $u: Q\to\mathbb{R}\}$,

$\mathrm{LSC}(Q) = \{$lower semi-continuous functions $u: Q\to\mathbb{R}\}$,

$\mathrm{C}(Q) = \{\text{continuous functions } u : Q \to \mathbb{R}\}$.

Definition 2.8.3. Let $v \in \mathrm{USC}(\mathbb{R} \times (0,\infty))$ (resp. $v \in \mathrm{LSC}(\mathbb{R} \times (0,\infty))$). We say that v is a **viscosity sub-solution** (resp. **super-solution**) of equation (2.8.3) on $\mathbb{R} \times (0,\infty)$ if for any point (x_0, t_0) with $t_0 > 0$ and any $\tau \in (0, t_0)$ and any test function $\varphi \in C^2(\mathbb{R} \times (0,\infty)) \cap L^\infty(\mathbb{R} \times (0,\infty))$ such that $v - \varphi$ attains a global maximum (minimum) at the point (x_0, t_0) on

$$Q_\tau = \mathbb{R} \times (t_0 - \tau, t_0]$$

we have that

$$\partial_t \varphi(x_0, t_0) + |\varphi_x(x_0, t_0)|^{m-1}((-\Delta)^\alpha \varphi(\cdot, t_0))(x_0) \le 0 \quad (\ge 0).$$

Since equation (2.8.3) is invariant under translation, the test function φ in the above definition can be taken such that φ touches v from above in the sub-solution case, resp. φ touches v from below in the super-solution case.

We say that v is a **viscosity sub-solution** (resp. **super-solution**) of the initial-value problem (2.8.3)-(2.8.4) on $\mathbb{R} \times (0,\infty)$ if it satisfies moreover at $t = 0$

$$v(x,0) \le \limsup_{y \to x,\ t \to 0} v(y,t) \quad (\text{resp. } v(x,0) \ge \liminf_{y \to x,\ t \to 0} v(y,t)).$$

We say that $v \in C(\mathbb{R} \times (0,\infty))$ is a **viscosity solution** if v is a viscosity sub-solution and a viscosity super-solution on $\mathbb{R} \times (0,\infty)$.

Proposition 2.8.4 (Existence of viscosity solutions)**.** *Let u be a weak solution for Problem (2.1.1). Then v defined by formula $v(x,t) = \int_{-\infty}^{x} u(y,t)dy$ is a viscosity solution for Problem (2.8.3)-(2.8.4).*

Proof. By Proposition 2.8.2 we know that $v \in C([0,T] : C(\mathbb{R}))$. The idea is to obtain a viscosity solution by the approximation process. Let v_ϵ defined by $v_\epsilon(x,t) = \int_{-\infty}^{x} u_\epsilon(y,t)dy$, where u_ϵ is the approximation of u as in Section 2.4. Then v_ϵ is a classical solution, in particular a viscosity solution, to the problem

$$(v_\epsilon)_t = \delta \Delta(v_\epsilon) + |(v_\epsilon)_x|^{m-1}(-\Delta)^{1-s} v_\epsilon.$$

Since $u_\epsilon \to u$, then we get that $v_\epsilon \to v$ as $\epsilon \to 0$ (and similarly with respect to the other parameters). The final argument is to prove that a limit of viscosity solutions is a viscosity solution of Problem (2.8.3)-(2.8.4).

□

The standard comparison principle for viscosity solutions holds true. We refer to Imbert, Monneau and Rouy [57] where they treat the case $m = 2$ and $\alpha = 1/2$. Also, we mention Jakobsen and Karlsen [58] for the elliptic case.

Proposition 2.8.5 (Comparison Principle)**.** *Let $m \in (1, 2)$, $\alpha \in (0, 1)$, $N = 1$. Let w be a sub-solution and W be a super-solution in the viscosity sense of equation* (2.8.3)*. If $w(x, 0) \le v_0 \le W(x, 0)$, then $w \le W$ in $\mathbb{R} \times (0, \infty)$.*

We give now our extended version of parabolic comparison principle, which represents an important instrument when using barrier methods. This type of result is motivated by the nonlocal character of the problem and the construction of lower barriers in a desired region $\Omega \subset \mathbb{R}$ possibly unbounded. This determines the parabolic boundary of a domain of the form $\Omega \times [0, T]$ to be $(\mathbb{R} \setminus \Omega) \times [0, T] \cup \mathbb{R} \times \{0\}$, where $\Omega \subset \mathbb{R}$. A similar parabolic comparison has been proved in [20] and has been used for instance in [20, 72].

Proposition 2.8.6. *Let $m > 1$, $\alpha \in (0, 1)$. Let v be a viscosity solution of Problem* (2.8.3)-(2.8.4)*. Let $\Phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that $\Phi \in C^2(\Omega \times (0, T))$. Assume that*

- $\Phi_t + |\Phi_x|^{m-1}(-\Delta)^\alpha \Phi < 0$ *for $x \in \Omega$, $t \in [0, T]$;*
- $\Phi(x, 0) < v(x, 0)$ *for all $x \in \mathbb{R}$ (comparison at initial time);*
- $\Phi(x, t) < v(x, t)$ *for all $x \in \mathbb{R} \setminus \Omega$ and $t \in (0, T)$ (comparison on the parabolic boundary).*

Then $\Phi(x, t) \le v(x, t)$ for all $x \in \mathbb{R}$, $t \in (0, T)$.

Proof. The proof relies on the study of the difference $\Phi - v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$. At the initial time $t = 0$ we have by hypothesis that $\Phi(x, 0) - v(x, 0) < 0$ for all $x \in \mathbb{R}$.

Now, we argue by contradiction. We assume that the function $\Phi - v$ has a first contact point (x_c, t_c) where $x_c \in \Omega$ and $t_c \in (0, T)$. That is, $(\Phi - v)(x_c, t_c) = 0$ and $(\Phi - v)(x, t) < 0$ for all $0 < t < t_c$, $x \in \mathbb{R}$, by regularity assumptions. Therefore, $(\Phi - v)$ has a global maximum point at (x_c, t_c) on $\mathbb{R} \times (0, t_c]$. Therefore, $v - \Phi$ attains a global minimum at (x_c, t_c).

Since v is a viscosity solution and Φ is an admissible test function then by definition

$$\Phi_t(x_c, t_c) + |\Phi_x(x_c, t_c)|^{m-1}(-\Delta)^\alpha \Phi(x_c, t_c) \ge 0,$$

which is a contradiction since this value is negative by hypothesis. □

2.8.4 Self-Similar Solutions. Formal approach

Self-similar solutions are the key tool in describing the asymptotic behaviour of the solution to certain parabolic problems. We perform here a formal computation of a type of self-similar solution to equation (2.8.3), being motivated by the construction of suitable lower barriers.

Let $m \in (1, 2)$ and $\alpha \in (0, 1)$. We search for self-similar solutions to equation (2.8.3) of the form

$$U(x, t) = \Phi(|y|t^{-b})$$

which solve equation (2.8.3) in $\mathbb{R} \times (0, \infty)$. After a formal computation, it follows that the exponent $b > 0$ is given by $b = 1/(m-1+2\alpha)$ and the profile function Φ is a solution of the equation

$$by\Phi'(y) - |\Phi'(y)|^{m-1}(-\Delta)^{\alpha}\Phi(y) = 0.$$

We deduce that any possible behavior of the form $\Phi(y) = c|y|^{-\gamma}$ with $\gamma > 1$ is given by

$$\gamma = \frac{2\alpha + m}{2 - m}. \tag{2.8.6}$$

The value of the self-similarity exponent will be used in the next section for the construction of a lower barrier. A further analysis of self-similar solutions is beyond the purpose of this chapter and can be the subject of a new work. We mention that in the case $m = 2$, the profile function Φ has been computed explicitly by Biler, Karch and Monneau in [13].

2.8.5 Construction of the lower barrier

In this section we present a class of sub-solutions of equation (2.8.3) which represent an important tool in the proof of the infinite speed of propagation. For a suitable choice of parameters this type of sub-solution will give us a lower bound for v in the corresponding domain. This motivates us to refer to this function as a lower barrier. We mention that a similar lower barrier has been constructed in [72].

Let $\gamma = \dfrac{m + 2\alpha}{2 - m}$ and $b = \dfrac{1}{m - 1 + 2\alpha}$ be the exponents deduced in Section 2.8.4.

We fix $x_0 < 0$. In the sequel we will use as an important tool a function $G : \mathbb{R} \to \mathbb{R}$ such that, given any two constants $C_1 > 0$ and $C_2 > 0$, we have that

- (G1) G is compactly supported in the interval $(-x_0, \infty)$;
- (G2) $G(x) \le C_1$ for all $x \in \mathbb{R}$;

- (G3) $(-\Delta)^s G(x) \le -C_2|x|^{-(1+2s)}$ for all $x < x_0$.

This technical result will be proven in Lemma 2.9.1 of Section 2.9 (Appendix).

Lemma 2.8.7 (**Lower Barrier**). *Let $x_0 < 0$, $\epsilon > 0$ and $\xi > 0$. Also, let G be a function with the properties (G1),(G2) and (G3). We consider the barrier*

$$\Phi_\epsilon(x,t) = (t+\tau)^{b\gamma}\left((|x|+\xi)^{-\gamma} + G(x)\right) - \epsilon, \quad t \ge 0,\ x \in \mathbb{R}. \tag{2.8.7}$$

Then for a suitable choice of the parameter $C_2 > 0$, the function Φ_ϵ satisfies

$$(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1}(-\Delta)^\alpha \Phi_\epsilon \le 0 \quad \text{for } x < x_0,\ t > 0. \tag{2.8.8}$$

Moreover, C_1 is a free parameter and $C_2 = C_2(N, m, \alpha, \tau)$.

Proof. We start by checking under which conditions Φ_ϵ satisfies (2.8.8), that is, Φ is a classical sub-solution of equation (2.8.3) in Q. To this aim, we have that

$$\begin{aligned}(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1}(-\Delta)^\alpha\Phi_\epsilon &= b\gamma\frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} + \gamma^{m-1}\frac{(t+\tau)^{b\gamma(m-1)}}{(|x|+\xi)^{(\gamma+1)(m-1)}}(-\Delta)^\alpha\Phi_\epsilon(x,t)\\ = b\gamma\frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} &+ \gamma^{m-1}\frac{(t+\tau)^{b\gamma(m-1)}}{(|x|+\xi)^{(\gamma+1)(m-1)}}(t+\tau)^{b\gamma}\Big((-\Delta)^\alpha[(|x|+\xi)^{-\gamma}] + (-\Delta)^\alpha G\Big).\end{aligned}$$

Now, by Lemma 2.9.2 we get the estimate $(-\Delta)^\alpha\left((|x|+\xi)^{-\gamma}\right) \le C_3|x|^{-(1+2\alpha)}$ for all $|x| \ge |x_0|$, with positive constant $C_3 = C_3(N,m,\alpha)$. At this step, we choose the parameter C_2 in the assumption (G2) to be at least $C_2 > C_3$. The precise choice will be deduced later. Since $\gamma = (\gamma+1)(m-1)+1+2\alpha$, we continue as follows:

$$\begin{aligned}&(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1}(-\Delta)^\alpha\Phi_\epsilon\\ &\quad\le b\gamma\frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} + \gamma^{m-1}\frac{(t+\tau)^{b\gamma m}}{(|x|+\xi)^{(\gamma+1)(m-1)}}(C_3-C_2)|x|^{-(1+2\alpha)}\\ &\quad= (|x|+\xi)^{-(\gamma+1)(m-1)}\cdot\\ &\qquad\cdot\Big(b\gamma(t+\tau)^{b\gamma-1}(|x|+\xi)^{-(1+2\alpha)} + \gamma^{m-1}(t+\tau)^{b\gamma m}(C_3-C_2)|x|^{-(1+2\alpha)}\Big)\\ &\quad\le (|x|+\xi)^{-(\gamma+1)(m-1)}|x|^{-(1+2\alpha)}\Big(b\gamma(t+\tau)^{b\gamma-1} + \gamma^{m-1}(t+\tau)^{b\gamma m}(C_3-C_2)\Big)\end{aligned}$$

which is negative for all $(x,t) \in Q$, if we ensure that C_2 is such that:

$$C_2 > C_3 + b\gamma^{2-m}\tau^{b\gamma(1-m)-1}. \tag{2.8.9}$$

This choice of C_2 is independent on the parameters $\xi,\ \epsilon$. ∎

From now on, we will take $\tau = 1$, which will be enough for our purpose. We can now prove the main result for the model (2.8.3) which in particular implies the infinite speed of propagation of model (2.1.1) for $1 < m < 2$ in dimension $N = 1$.

2.8.6 Proof Theorem 2.8.1

Let $x_0 < 0$ fixed. We prove that $v(x,t) > 0$ for all $t > 0$ and $x < x_0$. By scaling arguments, the initial data v_0 with properties (2.8.5), satisfies

$$v_0(x) \geq H_{x_0}(x) = \begin{cases} 0, & x < x_0, \\ 1, & x > x_0. \end{cases} \tag{2.8.10}$$

We will prove that $v(x,t) \geq \Phi_\epsilon(x,t)$ in the parabolic domain $Q_T = \{x < x_0,\ t \in [0,T]\}$ by using as an essential tool the Parabolic Comparison Principle established in Proposition 2.8.6. We describe the proof in the graphics below, where the barrier function is represented, for simplicity, without the modification caused by the function $G(\cdot)$ (See Figure 2.2 and Figure 2.3).

To this aim we check the required conditions in order to apply the above mentioned comparison result.

• **Comparison on the parabolic boundary.** This will be done in two steps.

(a) **Comparison at the initial time.** The initial data (2.8.10) naturally impose the following conditions on Φ_ϵ. At time $t = 0$ we have $\Phi_\epsilon(x_0, 0) < 0$, which holds only if ξ satisfies

$$\xi > x_0 + \epsilon^{-\frac{1}{\gamma}}. \tag{2.8.11}$$

Therefore $\Phi_\epsilon(x_0, 0) < v_0(x_0)$ since $v_0(x_0) > 0$.

(b) **Comparison on the lateral boundary.** Let $k_1 := \min\{v(x,t) : \ x \geq x_0,\ 0 < t \leq T\}$ with $k_1 > 0$. This results follows from the continuity $v \in C([0,T] : C(\mathbb{R}))$ since $v_0(x_0) = 1$. We impose the condition

$$\Phi_\epsilon(x,t) < v(x,t) \quad \text{ for all } x \geq x_0,\ t \in [0,T].$$

It is sufficient to have

$$(T+1)^{b\gamma}(\xi^{-\gamma} + C_1) < k_1.$$

The maximum value of T for which this inequality holds is

$$T < \left(\frac{k_1}{\xi^{-\gamma} + C_1}\right)^{1/b\gamma} - 1. \tag{2.8.12}$$

We need to impose a compatibility condition on the parameters in order to have $T > 0$, that is:

$$\xi > (k_1 - C_1)^{-\frac{1}{\gamma}}. \tag{2.8.13}$$

The remaining parameter C_1 from assumption (G2) is chosen here such that: $C_1 < k_1$.

By Proposition 2.8.6 we obtain the desired comparison

$$v(x,t) \geq \Phi_\epsilon(x,t) \quad \text{for all } (x,t) \in Q_T.$$

• **Infinite speed of propagation**. Let $x_1 < x_0$ and $t_1 \in (0,T)$ where T is given by (2.8.12). We prove there exists a suitable choice of ξ and ϵ such that $\Phi_\epsilon(x_1,t_1) > 0$. This is equivalent to impose the following upper bound on ξ:

$$\xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}}. \tag{2.8.14}$$

We need to check now if there exists $\epsilon > 0$ such that condition (2.8.14) is compatible with conditions (2.8.11) and (2.8.13). For the compatibility of conditions (2.8.11) and (2.8.13) we have

$$x_0 + \epsilon^{-\frac{1}{\gamma}} < \xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}},$$

that is,

$$\epsilon < \left[\frac{(t_1+1)^b - 1}{x_0 - x_1}\right]^\gamma. \tag{2.8.15}$$

For conditions (2.8.13) and (2.8.14) we need

$$(k_1 - C_1)^{-\frac{1}{\gamma}} \leq \xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}},$$

which is equivalent to

$$\epsilon < \left[\frac{(t_1+1)^b}{(k_1 - C_1)^{-\frac{1}{\gamma}} - x_1}\right]^\gamma. \tag{2.8.16}$$

Both upper bounds (2.8.15) and (2.8.16) make sense since $0 > x_0 > x_1$ and $k_1 > C_1$.

Summary. The proof was performed in a constructive manner and we summarize it as follows: $C_1 < k_1$, T given by (2.8.12). Then by taking ϵ the minimum of (2.8.15)-(2.8.16), ξ satisfying (2.8.11)-(2.8.13)-(2.8.14) we obtain that $\Phi(t_1,x_1) > 0$.

This proofs that $v(t_1,x_1) > 0$ for any $t \in (0,T)$.

□

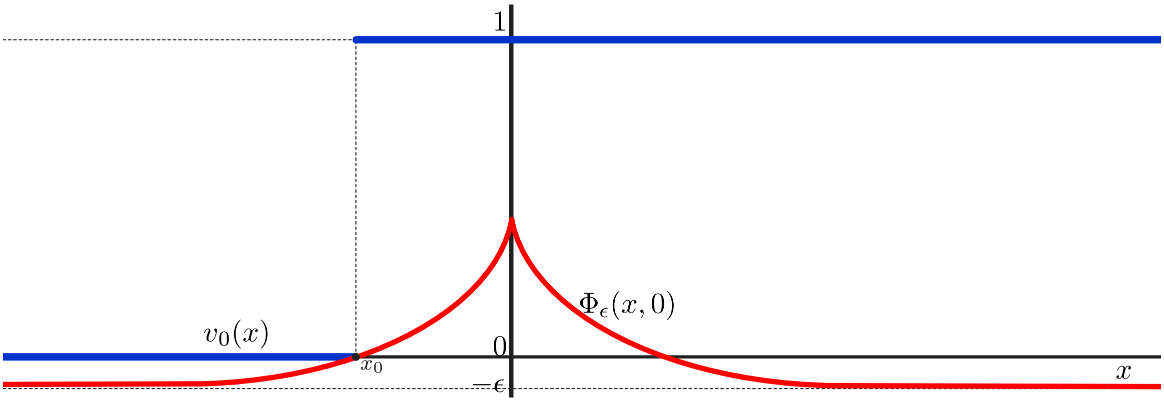

FIGURE 2.2: Comparison with the barrier at time $t = 0$

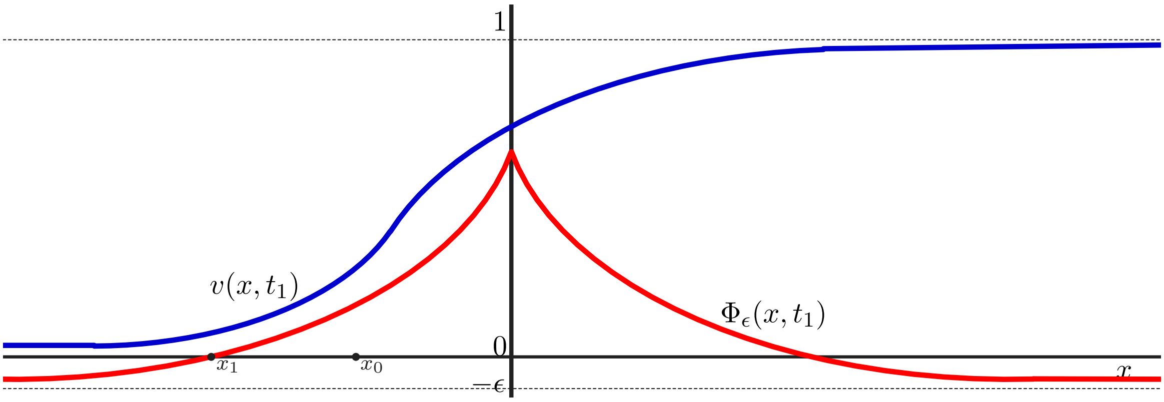

FIGURE 2.3: Comparison with the barrier at time $t > 0$

Remark. The parameter ξ of the barrier depends on ϵ by (2.8.11) and (2.8.14) and therefore $\xi \to \infty$ as $\epsilon \to 0$. Therefore $\Phi_\epsilon(x,t) \to 0$ as $\epsilon \to 0$ for every $(x,t) \in Q_T$ and we can not derive a lower parabolic estimate for $v(x,t)$ in Q_T.

2.9 Appendix

2.9.1 Estimating the Fractional Laplacian

In this section we are interested in estimating the fractional Laplacian of given functions. We recall the definition of the Fractional Laplacian operator

$$(-\Delta)^s u(x) = \sigma_{N,s}\ P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}, \quad 0 < s < 1,$$

where σ_s a normalization constant given by

$$\sigma_{N,s} = \frac{2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{N/2}\Gamma(-N/2)}.$$

First, given the expression of the fractional Laplacian, we construct a function with the desired properties.

Lemma 2.9.1. *Given two arbitrary constants $C_1, C_2 > 0$ there exists a function $G : \mathbb{R} \to [0, +\infty)$ with the following properties:*

1. *G is compactly supported.*
2. *$G(x) \le C_1$ for all $x \in \mathbb{R}$*
3. *$(-\Delta)^s G(x) \le -C_2 |x|^{-(1+2s)}$ for all $x \in \mathbb{R}$ with $d(x, supp(G)) \ge 1$.*

Proof. Let R an arbitrary positive number to be chosen later. We consider a smooth function $G_1 : \mathbb{R} \to [0, +\infty)$ such that $G_1(x) \le C_1$ for all $x \in \mathbb{R}$ and supported in the interval $[-1, 1]$.

We define $G_R(x) = G_1(x/R)$. Therefore $\|G_R\|_{L^1(\mathbb{R})} = R\|G_1\|_{L^1(\mathbb{R})}$, $G_R \le C_1$ and G is supported in the interval $[-R, R]$. Then for $|x| \ge R + 1$ we have that

$$\begin{aligned}
(-\Delta)^s G_R(x) &= \sigma_s \int_{\mathbb{R}} \frac{G_R(x) - G_R(y)}{|x-y|^{1+2s}} dy = -\sigma_s \int_{-R}^{R} \frac{G_R(y)}{|x-y|^{1+2s}} dy \\
&\le -\sigma_s \int_{-R}^{R} \frac{G_R(y)}{(|x| + R)^{1+2s}} dy = -\sigma_s (|x| + R)^{-(1+2s)} \|G_R\|_{L^1(\mathbb{R})} \\
&\le -\sigma_s 2^{-(1+2s)} \|G_R\|_{L^1(\mathbb{R})} |x|^{-(1+2s)} = -\sigma_s 2^{-(1+2s)} R \|G_1\|_{L^1(\mathbb{R})} |x|^{-(1+2s)}.
\end{aligned}$$

It is enough to choose $R \ge \dfrac{C_2 2^{1+2s}}{\sigma_s ||G_1||_{L^1(\mathbb{R})}}$ to get $(-\Delta)^s G_R(x) \le C_2 |x|^{-(1+2s)}$. Note that R implicitly depends on C_1 since $||G_1||_{L^1(\mathbb{R})} \le 2C_1$.

□

Secondly, we need to estimate the fractional Laplacian of a negative power function. The following result is similar to one proven by Bonforte and Vázquez in Lemma 2.1 from [17] with the main difference that our function is C^2 away from the origin. We make a brief adaptation of their proof to our situation.

Lemma 2.9.2. *Let $\varphi : \mathbb{R} \to (0, \infty)$, $\varphi = (|x| + \xi)^{-\gamma}$, where $\gamma > 1$ and $\xi > 0$. Then, for all $|x| \ge |x_0| > 1$, we have that*

$$|(-\Delta)^s \varphi(x)| \le \frac{C}{|x|^{1+2s}}, \tag{2.9.1}$$

with positive constant $C > 0$ that depends only on γ, ξ, s.

Proof. Let us first estimate the L^1 norm of φ.

$$\int_{\mathbb{R}} \varphi(x)dx = \int_{|x|<1} \varphi(x)dx + \int_{|x|>1} \varphi(x)dx \leq \int_{|x|<1} \xi^{-\gamma}dx + \int_{|x|>1} x^{-\gamma}dx$$
$$\leq 2\xi^{-\gamma} + 2\int_1^\infty r^{-\gamma}dr = 2\xi^{-\gamma} + \frac{2}{\gamma-1} < C, \quad C = C(\gamma,\xi).$$

Following the ideas of [17] Lemma 2.1, the computation of the $(-\Delta)^s\varphi(x)$ is based on estimating the integrals on the regions

$$R_1 = \{y: \ |y| > 3|x|/2\}, \quad R_2 = \left\{y: \ \frac{|x|}{2} < |y| < \frac{3|x|}{2}\right\} \setminus B_{\frac{|x|}{2}}(x),$$

$$R_3 = \{y: \ |x-y| < |x|/2\}, \quad R_4 = \{y: \ |y| < |x|/2\}.$$

Therefore

$$(-\Delta)^s\varphi(x) = \int_{R_1\cup R_2\cup R_3\cup R_4} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy.$$

We proceed with the estimate of each of the four integrals:

$$I = \int_{|y|>3|x|/2} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \leq \omega_d\varphi(x)\int_{3|x|/2}^\infty \frac{dr}{r^{1+2s}} = \frac{K_1}{|x|^{\gamma+2s}}, \quad K_1 = K_1(\gamma,s).$$

$$II = \int_{R_2} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \leq \frac{\varphi(x)}{(|x|/2)^{1+2s}}\int_{|x|/2}^{3|x|/2} dr = \frac{K_2}{|x|^{\gamma+2s}}, \quad K_2 = K_2(\gamma,s).$$

$$III = \int_{R_3} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \leq \|\varphi''\|_{L^\infty(B_{|x|/2}(x))}\int_{|x-y|\leq|x|/2} \frac{1}{|x-y|^{2s-1}}dy$$
$$\leq \frac{K_3'}{|x|^{\gamma+2}}\int_0^{|x|/2} \frac{1}{r^{2s-1}}dr \leq \frac{K_3}{|x|^{\gamma+2s}}, \qquad K_3 = K_3(\gamma,s).$$

For IV we take use that when $|y| < |x|/2$ then $|y-x| \geq |x|/2$ and $|y| < |x|$ which implies $\varphi(y) > \varphi(x)$. We have

$$IV \leq \int_{|y|<|x|/2} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{1+2s}}dy \leq \left(\frac{2}{|x|}\right)^{1+2s}\int_{|y|<|x|/2}\varphi(y)dy \leq \left(\frac{2}{|x|}\right)^{1+2s}\|\varphi\|_{L^1(\mathbb{R})}$$
$$\leq \frac{K_4}{|x|^{1+2s}}, \quad K_4 = K_4(\gamma,s,\xi).$$

Since $\gamma > 1$, we can conclude that

$$|(-\Delta)^s\varphi(x)| \leq |I|+|II|+|III|+|IV| = K_5|x|^{-\gamma-2s}+K_4|x|^{-1-2s} \leq K_6|x|^{-1-2s}, \quad \forall|x|\geq|x_0|>1.$$

$\square$

2.9.2 Reminder on cut-off functions

We remind the construction of cut-off functions. Let

$$f(x) = \begin{cases} e^{-1/x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$. Let

$$F(x) = \frac{f(x)}{f(x) + f(1-x)}, \quad x \in \mathbb{R}.$$

Then $F(x) = 0$ for $x < 0$, $F(x) = 1$ for $x \geq 1$ and $F(x) \in (0,1)$ for $x \in (0,1)$. We construct now the cut-off function $\varphi : \mathbb{R}^N \to [0,1]$ by:

$$\varphi(x) = F(2 - |x|),\ \overline{x} \in \mathbb{R}^N.$$

Then $\varphi \in C^\infty(\mathbb{R}^N)$, $\varphi(x) = 1$ for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$ and $\varphi(x) \in (0,1)$ for $|x| \in (1,2)$. The cut-off function for B_R is obtained by

$$\varphi_R(x) = \varphi(x/R).$$

Thus $\varphi_R \in C^\infty(\mathbb{R}^N)$, $\varphi_R(x) = 1$ for $|x| \leq R$, $\varphi_R(x) = 0$ for $|x| \geq 2R$ and $\varphi_R(x) \in (0,1)$ for $|x| \in (R, 2R)$. Also, we have that $\nabla(\varphi_R) = O(R^{-1})$, $\Delta(\varphi_R) = O(R^{-2})$.

2.9.3 Compact sets in the space $L^p(0,T;B)$

Necessary and sufficient conditions of convergence in the spaces $L^p(0,T;B)$ are given by Simon in [67]. We recall now their applications to evolution problems. We consider the spaces $X \subset B \subset Y$ with compact embedding $X \to B$.

Lemma 2.9.3. *Let $\mathcal{F}$ be a bounded family of functions in $L^p(0,T;X)$, where $1 \leq p < \infty$ and $\partial\mathcal{F}/\partial t = \{\partial f/\partial t : f \in \mathcal{F}\}$ be bounded in $L^1(0,T;Y)$. Then the family $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$.*

Lemma 2.9.4. *Let $\mathcal{F}$ be a bounded family of functions in $L^\infty(0,T;X)$ and $\partial\mathcal{F}/\partial t$ be bounded in $L^r(0,T;Y)$, where $r > 1$. Then the family $\mathcal{F}$ is relatively compact in $C(0,T;B)$.*

2.10 Comments and open problems

• **Case $m \geq 3$.** In this range of exponents the first energy estimate does not hold anymore. Therefore, we lose the compactness result needed to pass to the limit in the approximations to obtain a weak solution of the original problem. The second energy estimate is still true and it gives us partial results for compactness. In our opinion a suitable tool to replace the first energy estimate would be proving the decay of some L^p norm. In that case we will also need a Stroock-Varoupolous type inequality for some approximation $\mathcal{L}_s^\epsilon$ of the fractional Laplacian. The technique of regularizing the kernel by convolution that we have used through this chapter does not allow us to prove such kind of inequality. The idea is however to use a different approximation of the pressure term that is well suited to the Stroock-Varoupolous type inequalities. Let us mention [46] where this kind of inequalities are proved for a wider class of nonlocal operators including $\mathcal{L}_s^\epsilon$. The technical details are involved and the new approximation may have an interest, hence we think it deserves a separate study.

• **Infinite propagation in higher dimensions for self similar solutions.** In [71] we proved a transformation formula between self-similar solutions of the model (2.1.1) with $1 < m < 2$ and the fractional porous medium equation $u_t + (-\Delta)^s u^m = 0$. This way we obtain infinite propagation for self similar solutions of the form $U(x,t) = t^{-\alpha} F(|x| t^{-\alpha/N})$ in $\mathbb{R}^N$. This is a partial confirmation that the property of the infinite speed of propagation holds in higher dimensions for every solution of (2.1.1) with $1 < m < 2$.

• **Explicit solutions.** Y. Huang reports [53] the explicit expression of the Barenblatt solution for the special value of m, $m_{ex} = (N + 6s - 2)/(N + 2s)$. The profile is given by

$$F_M(y) = \lambda \, (R^2 + |y|^2)^{-(N+2s)/2},$$

where the two constants λ and R are determined by the total mass M of the solution and the parameter β. Note that for $s = 1/2$ we have $m_{ex} = 1$, and the solution corresponds to the linear case, $u_t = (-\Delta)^{1/2} u$, $F_{1/2}(r) = C(a^2 + r^2)^{-(N+1)/2}$.

• **Different generalizations of model** (2.1.2) are worth studying:

(i) Changing-sign solutions for the problem $\partial_t u = \nabla \cdot (|u| \nabla p), \quad p = (-\Delta)^{-s} u$.

(ii) Starting from the Problem (2.1.2), an alternative is to consider the problem

$$u_t = \nabla \cdot (|u| \nabla (-\Delta)^{-s} (|u|^{m-2} u)), \quad x \in \mathbb{R}^N, \ t > 0,$$

with $m > 1$. This problem has been studied by Biler, Imbert and Karch in [12]. They construct explicit compactly supported self-similar solutions which generalize the Barenblatt profiles of the PME. In a later work by Imbert [56], finite speed of propagation is proved for general solutions.

(iii) We should consider combining the above models into $\partial_t u = \nabla(|u|^{m-1}\nabla p)$, $p = (-\Delta)^{-s}u$.
When $s = 0$ and $m = 2$ we obtain the signed porous medium equation $\partial_t u = \Delta(|u|^{m-1}u)$.

Chapter 3

Transformations of self-similar solutions of nonlocal porous medium equations

3.1 Introduction

We will consider the following models of evolution equations of diffusive type involving at the same type fractional Laplacian operators and power nonlinearities:

$$u_t + (-\Delta)^s u^m = 0, \tag{M1}$$

$$v_t = \nabla \cdot (v^{\widetilde{m}-1} \nabla (-\Delta)^{-\widetilde{s}} v), \tag{M2}$$

$$w_t = \nabla \cdot (w \nabla (-\Delta)^{-\widehat{s}} w^{\widehat{m}-1}), \tag{M3}$$

as well as the more general version

$$z_t = \nabla (z^{\mathring{m}-1} \nabla (-\Delta)^{-\mathring{s}} z^{\mathring{n}-1}). \tag{MG}$$

For brevity we will call them (M1), (M2), (M3) and (MG), where M stands for Model and G for General. Here, $(-\Delta)^s$ is the fractional Laplacian operator, $0 < s < 1$, with Fourier symbol $|\xi|^{2s}$, and $(-\Delta)^{-s}$ its inverse, cf. [63, 73]. Due to the power nonlinearities we refer to them as fractional diffusion equations of porous medium type. See details about current research concerning Model 1 in [17, 37, 38], cf. [69, 70] for Model 2, and [12, 13] for Model 3, as well as the survey papers [78, 81]. When $\widetilde{m} = \hat{m} = 2$, then (M2) and (M3) coincide and the problem becomes $v_t = \nabla \cdot (v \nabla (-\Delta)^{-\widetilde{s}} v)$. Existence, finite propagation and self-similarity for this problem were recently studied in [22, 24, 25].

The behavior of the solutions of the three basic models may be very different depending both on the equation and on the parameters $m, \widetilde{m}$ and $\widehat{m}$. An efficient way of studying such differences is via the existence and properties of special solutions having particular symmetries since such solutions are either explicit or semi-explicit, or at least can be analyzed in great detail; the interest is also due to the fact that they are important in describing the properties of much wider classes of solutions. This applies in particular to the class of self-similar solutions, namely, solutions of the types

$$u(x,t) = t^{-\alpha}\phi(xt^{-\beta})$$

(so-called type I), or

$$u(x,t) = (T-t)^{\alpha}\phi(x\,(T-t)^{-\beta})$$

(type II). The importance of self-similar solutions in the areas of PDEs and Applied Mathematics is attested in a wide literature, cf. Barenblatt's monograph [8] or [76].

The present chapter is concerned with the existence, properties and correspondences of self-similar solutions of the four fractional diffusion models presented above. Important progress has been done recently in the study of self-similar solutions of these models. The questions of existence and properties are not an easy task since in principle the solutions are not explicit. Important questions in the qualitative analysis are the following: (i) the equation satisfied by the profile function, (ii) the decay of the profile, (iii) whether or not there is an explicit expression for ϕ, (iv) whether the profile is compactly supported or not (finite versus infinite propagation); (v) a crucial question is the relation of these solutions to the general theory, in particular whether the large-time behavior of a general solution of the equation is given by a self-similar solution. These are difficult questions and there are only partial answers in the literature for some of the models.

Here we investigate the existence of transformations that enable to pass from self-similar solutions of one of the equations into self-similar solutions of another equation, thus showing some deep connection between the models, and transferring results from one model to another one. Several coincidences had been observed in the recent literature in particular cases, see Biler et al. [12], Huang [53], Vázquez [80]. We will show below that there are general transformations that apply to important classes of self-similar solutions of our models, putting whole ranges of self-similar solutions in one-to-one correspondence. We will devote special attention to the correspondence between models (M1) and (M2).

Transformations between whole classes of solutions of different equations can be very useful but they are not frequent in the literature. There are however some well-known examples. Let us mention some of them in the area of nonlinear diffusion: (i) the

Hopf-Cole transformation that maps solutions of the heat equation into solutions of Burgers equation [35, 52]; (ii) the Lie-Bäcklund transformation that maps solutions of the fast diffusion equation into solutions of the porous medium equation in 1D, [15]; (iii) differentiation in space maps solutions of the p-Laplacian equation in 1D into solutions of the porous medium equation; (iv) the relationship between these equations has been extended to the whole class of radial solutions in several dimensions by Iagar, Sánchez and Vázquez in [55].

In Section 3.6 we find explicit very singular solutions for model (M2) of two types, in a separate variables form. The first type are solutions which are positive for all times for some values of $\widetilde{m}$ in the supercritical range $\widetilde{m} > (N-2+2\widetilde{s})/N$, while the second type are solutions that extinguish in finite time for all $\widetilde{m}$ in the subcritical range $0 < \widetilde{m} < (N-2+2\widetilde{s})/N$. Both types of solutions have a form algebraically similar to the ones found by Vázquez in [79] and by Vázquez and Volzone in [84] for model M1.

3.2 Preliminaries on Model (M1)

This problem is probably the best known of the list. The equation is called the *Fractional Porous Equation*, FPMEchap3, since it can be considered as the fractional version of the standard Porous Medium Equation $u_t = \Delta u^m$. On the other hand, for $m = 1$ and $0 < s < 1$ we get the linear fractional heat equation, which has been also well studied.

The existence, uniqueness and continuous dependence of solutions of the Cauchy problem (M1) for all $m > 0$ and $0 < s < 1$ have been proved by De Pablo, Quirós, Rodríguez and Vázquez in [37, 38]. The main result of interest here is the property of infinite speed of propagation: Assume $s \in (0,1)$, $m > m_c = (N-2s)_+/N$. Then for non-negative initial data $u_0 \geq 0$, $\int_{\mathbb{R}^N} u_0(x)\,dx < \infty$, there exists a unique solution $u(x,t)$ of problem (M1) satisfying $u(x,t) > 0$ for all $x \in \mathbb{R}^N$, $t > 0$. Moreover, mass that is conserved, $\int u(x,t)\,dx = \int u_0(x)\,dx$ for all $t > 0$.

The long term behavior of such solutions is described by the self-similar solutions of type I with finite mass (Barenblatt solutions) constructed in [79], where it is shown that the equation admits a family of self-similar solutions, also called Barenblatt type solutions, of the form

$$u(x,t) = t^{-N\beta_1}\phi_1(y), \quad y = x\,t^{-\beta_1}\,,$$

and $\beta_1 = 1/(N(m-1)+2s)$. Existence is proved when $m > m_c$, so that β_1 is well-defined and positive. The extra condition that is used to obtain these solutions is $\int u(x,t)\,dx =$ constant in time. This formula produces a solution to equation (M1) if the profile function ϕ_1 satisfies the following equation

$$(-\Delta)^s \phi_1^m = \beta_1 \nabla \cdot (y\, \phi_1). \tag{3.2.1}$$

It is proved in the above reference that the profile $\phi_1(y)$ is a smooth and positive function in $\mathbb{R}^N$, it is a radial function, it is monotone decreasing in $r = |y|$ and has a definite decay rate as $|y| \to \infty$, that depends on m a bit, as described below.

Theorem 3.2.1. *For every choice of parameters $s \in (0,1)$ and $m > m_c$ and for every $M > 0$, equation (M1) admits a unique fundamental solution $u_M^*(x,t)$; it is a nonnegative and continuous weak solution for $t > 0$ and takes the initial data $M\,\delta(x)$ as a trace in the sense of Radon measures. It has the self-similar form of type I for suitable α and β that can be calculated in terms of N and s in a dimensional way, precisely*

$$\alpha = \frac{N}{N(m-1)+2s}, \qquad \beta = \frac{1}{N(m-1)+2s}. \tag{3.2.2}$$

The profile function $\phi_M(r)$, $r \geq 0$, is a bounded and continuous function, it is positive everywhere, it is monotone and it goes to zero at infinity.

Moreover, the precise characterization of the profile ϕ_1 is given by Theorem 8.1 of [79].

Theorem 3.2.2. *For every $m > m_1 = N/(N+2s)$ we have the asymptotic estimate*

$$\lim_{r\to\infty} \phi_1(r) r^{N+2s} = C_1 M^\sigma, \tag{3.2.3}$$

where $M = \int \phi_1(r)\, dx$, $C_1 = C_1(m,N,s) > 0$ and $\sigma = (m-m_1)(N+2s)\beta$. On the other hand, for $m_c < m < m_1$, there is a constant $C_\infty(m,N,s)$ such that

$$\lim_{r\to\infty} \phi_1(r) r^{2s/(1-m)} = C_\infty. \tag{3.2.4}$$

The case $m = m_1$ has a logarithmic correction. The profile ϕ_1 has the upper bound

$$\phi_1(r) \leq C r^{-N-2s+\epsilon}, \quad \forall r > 0 \tag{3.2.5}$$

for every $\epsilon > 0$, and the lower bound

$$\phi_1(r) \geq C r^{-N-2s} \log r, \qquad \text{for all large } r. \tag{3.2.6}$$

As a consequence, the asymptotic behavior of general solutions of the Cauchy Problem for equation (M1) is represented by such special solutions as described in Theorem 10.1 from [79].

Theorem 3.2.3. *Let $u_0 = \mu \in \mathcal{M}_+(\mathbb{R}^N)$, let $M = \mu(\mathbb{R}^N)$ and let B_M be the self-similar Barenblatt solution with mass M. Then we have*

$$\lim_{t\to\infty} |u(x,t) - B_M(x,t;M)| = 0 \tag{3.2.7}$$

and the convergence is uniform in $\mathbb{R}^N$.

More delicate properties of general solutions to problem (M1) have been proved recently: a priori estimates, quantitative bounds on positivity and Harnack estimates by Bonforte and Vázquez [17], a priori estimates derived by Schwartz symmetrization technique by Vázquez and Volzone in [83, 84], and numerical computations by Teso and Vázquez in [40, 41].

A main practical question that remains partially open is to determine if the profile ϕ_1 can be expressed as an explicit or semi-explicit function of $r = |x|$ (and the parameters s and N). The answer is yes in the special case $m = 1$ where the solution is explicit for $s = 1/2$, semi-explicit otherwise. Recently, Huang [53] has shown that for every $s \in (0,1)$ there exists a certain $m_{ex}(s) > m_1$ for which the profile has an explicit expression. More precisely, $m_{ex}(s) = (N+2-2s)/(N+2s)$. For $s = 1/2$ we have $m_{ex}(s) = 1$, thus recovering the formula of the linear fractional heat equation.

For $m < (N-2s)/N$, model (M2) admits self-similar solutions of type II, as proved by Vázquez and Volzone in [84]. Here we will prove an equivalence between (M1) with $m > (N-2s)/N$ and (M2) with a corresponding $\widetilde{m}$ interval.

3.3 Model (M2). First correspondence between models

3.3.1 Preliminaries on Model (M2)

We call the equation of (M2) the *Porous Medium Equation with Fractional Pressure* since it can be written as $u_t = \nabla(u^{\widetilde{m}-1}\nabla p)$ with the pressure $p = (-\Delta)^{-s}u$.

(i) The study of the problem has been done by Caffarelli and Vázquez [24, 25] and also with Soria [22] in the more natural case $\widetilde{m} = 2$. Previous analysis in 1D is due to Biler et al. [12]. It is proved that for non-negative initial data $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^N)$, there exists a non-negative solution $u(x,t) \geq 0$. However, uniqueness of the constructed weak solutions has not been proved but for the case $N = 1$. Moreover, the assumption of compact support on the initial data implies that the same property for all positive times, $u(\cdot,t)$ is compactly supported for all $t > 0$. The existence of a self-similar solution that will be responsible for the asymptotic behavior is obtained in [12] in 1D and in [24] in all dimensions as the solution of a fractional obstacle problem. The explicit formula for this solution was given in [13], and takes the form

$$v(x,t) = t^{-N//N+2-2s}\Phi(xt^{-1/(N+2-2s)}), \qquad \Phi(y) = (a - b|y|^2)_+^{1-s} \tag{3.3.1}$$

for suitable constants $a, b > 0$.

(ii) The present authors have extended these results for general $m > 1$ in [69] and the preprint [70]. In these recent works we prove that for non-negative initial data $u_0 \geq 0$, there exists a non-negative solution $u(x,t) \geq 0$. Different results on the positivity properties have been obtained depending on the parameter m as follows:

- When $N \geq 1$, $s \in (0,1)$, $u_0 \geq 0$ compactly supported and $\widetilde{m} \in [2,\infty)$ then the solution $u(x,t)$ is compact supported for all $t > 0$, that is the model has finite speed of propagation.

- When $N = 1$, $s \in (0,1)$, $\widetilde{m} \in (1,2)$ and $u_0 \geq 0$ then the solution satisfies $u(x,t) > 0$ a.e. in $\mathbb{R}$, therefore the model has infinite speed of propagation.

3.3.2 Self-similarity for Model (M2)

We find two main types of self-similar solutions for model (M2) depending on the range of the parameter $\widetilde{m}$. The first type are functions that are positive for all times, while second type are functions that extinguish in finite time, separated by a transition type.

• **Self-similarity of first type. Solutions that exist for all positive times.** Arguing in the same way as in Model 1, or the case $\widetilde{m} = 2$ of Model 2 described above, a self-similar function of the first type $v(x,t)$ is a solution to equation (M2) conserving mass if

$$v(x,t) = t^{-\alpha_2}\phi_2(y), \quad y = x\,t^{-\beta_2} \tag{3.3.2}$$

with $\alpha_2 = N\beta_2$ and $\beta_2 = 1/(N(\widetilde{m}-1) + 2 - 2\widetilde{s})$, and if the profile function ϕ_2 satisfies the equation

$$\nabla \cdot (\phi_2^{\widetilde{m}-1}\,\nabla(-\Delta)^{-\widetilde{s}}\phi_2) = -\beta_2 \nabla \cdot (y\,\phi_2). \tag{3.3.3}$$

The existence and properties of this family of solutions have not been studied in the literature, but for the work of Huang ([53]) who has shown the existence of a certain $m(s)$ for each $s \in (0,1)$ for which an explicit solution can be found.

Remark. In the analysis below we find these solutions in the range of parameters where $\beta_2 > 0$, that is, for $\widetilde{m} > (N - 2 + 2\widetilde{s})/N$.

• **Self-Similarity of second type. Extinction in finite time.** We will also search for solutions of the second self-similar form

$$v(x,t) = (T-t)^{\overline{\alpha}_2}\psi_2\,(y)\,, \quad y = x(T-t)^{\overline{\beta}_2}. \tag{3.3.4}$$

We make again the choice $\overline{\alpha}_2 = N\overline{\beta}_2$ even if there can be no justification in terms of mass conservation since the solutions will now extinguish in finite time (the solution to this seeming incompatibility is that the mass will be actually infinite). We use however the rule for a formal consideration: the divergence structure of the resulting profile equation will make the correspondence with Model (M1) possible.

Let us determine the value of $\overline{\beta}_2$ such that $v(x,t)$ solves the equation of (M2). Since

$$v_t(x,t) = -\overline{\beta}_2(T-t)^{N\overline{\beta}_2-1}\nabla\cdot(y\psi_2).$$

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v) = (T-t)^{\overline{\beta}_2(N\widetilde{m}-2\widetilde{s}+2)}\nabla\cdot(\psi_2^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}\psi_2),$$

we get the compatibility condition

$$\overline{\beta}_2 = 1/(N(1-\widetilde{m}) + 2\widetilde{s} - 2)\,.$$

The profile ψ_2 has to satisfy the equation

$$\nabla\cdot(\psi_2^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}\psi_2) = \nabla\cdot(y\psi_2). \tag{3.3.5}$$

Remark. $\overline{\beta}_2 = -\beta_2$, where β_2 is the self-similarity exponent of first type. We argue now in the range of parameters where $\overline{\beta}_2 > 0$, that is $\widetilde{m} < (N-2+2\widetilde{s})/N$.

• **Self-Similarity of third type. Eternal solutions.** There is a borderline case $\widetilde{m} = (N-2+2\widetilde{s})/N$, which is not included in the previous self-similar solutions. Actually, as $m \to (N-2+2\widetilde{s})/N$ we have $1/\beta_2 = 1/\overline{\beta}_2 \to 0$, and therefore self-similar solutions of the first and second type do not apply here. The possibility of self-similar representation comes from the classical porous medium equation (see [79]) where a third type of self-similar solutions of the form

$$v(x,t) = e^{-\gamma t}F(y), \quad y = xe^{-ct}. \tag{3.3.6}$$

where $c > 0$ is a free parameter (exponential self-similarity, which usually plays a transition role). We choose $\gamma = ct$ in order to have conservation of mass. It is easy to check that

$$v_t(x,t) = -ce^{Nct}\nabla\cdot(yF),$$

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v)=e^{-ct(-N\widetilde{m}+2\widetilde{s}-2)}\nabla\cdot(F^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}F).$$

Then, for $m=(N-2+2\widetilde{s})/N$ we get the following profile equation

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v)=-c\nabla\cdot(yF) \tag{3.3.7}$$

Remark. Solutions of this type live backwards and forward in time, they are eternal.

3.3.3 Equivalence relation

A main contribution in this chapter is to show a relationship that allows to transform the families of mass-conserving self-similar solutions of models (M1) and (M2) into each other, if suitable parameter ranges are prescribed. Actually, the following theorem states that there exists a precise correspondence between the profiles ϕ_1 and ϕ_2, and the parameters $\widetilde{m}$ and m, as well as $\widetilde{s}$ and s.

Theorem 3.3.1. *Let $m>(N-2s)/N$, $s\in(0,1)$ and let $\phi_1\geq 0$ be a solution to the profile equation (3.2.1). The following holds:*

(i) *If $m>N/(N+2s)$ then*

$$\phi_2(x)=(\beta_1/\beta_2)^{\frac{m}{1-m}}\,(\phi_1(x))^m \tag{3.3.8}$$

is a solution to the profile equation (3.3.3) if we put $\widetilde{m}=(2m-1)/m$ and $\widetilde{s}=1-s$.

(ii) *If $m\in((N-2s)/N,N/(N+2s))$ then*

$$\psi_2(x)=\left(\beta_1/\overline{\beta}_2\right)^{\frac{m}{1-m}}\,(\phi_1(x))^m \tag{3.3.9}$$

is a solution to the profile equation (3.3.5) if we put $\widetilde{m}=(2m-1)/m$ and $\widetilde{s}=1-s$.

(iii) *If $m=N/(N+2s)$ then*

$$F(x)=(\beta_1/c)^{\frac{N}{2s}}\,(\phi_1(x))^{\frac{N}{N+2s}} \tag{3.3.10}$$

is a solution to the profile equation (3.3.7) if we put $\widetilde{m}=(N-2+2\widetilde{s})/N$ and $\widetilde{s}=1-s$.

Comments. The first case corresponds to exponents β_1 and $\beta_2>0$ and produces new self-similar solutions of (M2) are type I, i. e., global in time. We see that $\beta_1>0$ if $m>(N-2s)/N$, while $\beta_2>0$ if $\widetilde{m}>(N-2+2\widetilde{s})/N$. With the relation $\widetilde{m}=(2m-1)/m$, we have $\widetilde{m}>(N-2+2\widetilde{s})/N$ which is equivalent to $m>N/(N+2s)$. This is another important value in the FPMEchap3, identified in [79], and we have

$N/(N+2s) > (N-2s)_+/N$. Therefore, by analyzing the parameters m and $\widetilde{m}$ for which $\beta_1 > 0$ and $\beta_2 > 0$ we have to work in the range of parameters $m > N/(N+2s)$.

(ii) This option produces solutions of (M2) that extinguish in finite time, starting with solutions of (M1) that exist globally in time. This is a remarkable phenomenon of change of behavior.

Proof. (1) Let us write equation (3.2.1) in terms of ϕ_2, that is, $\phi_1 = (\beta_2/\beta_1)^{\frac{1}{(1-m)}}\,\phi_2^{\frac{1}{m}}$, and then

$$(-\Delta)^s\phi_2 = \beta_2\nabla\cdot(y\,\phi_2^{\frac{1}{m}}).$$

Now, we pass to the parameters $\widetilde{m}$ and $\widetilde{s}$ defined by

$$m = \frac{1}{2-\widetilde{m}} \quad \text{and} \quad s = 1-\widetilde{s} \tag{3.3.11}$$

and we obtain

$$-\Delta(-\Delta)^{-\widetilde{s}}\phi_2 = \beta_2\,\nabla\cdot(y\,\phi_2^{2-\widetilde{m}}).$$

We can express now Δ as $\nabla\cdot\nabla$, integrate once and use the decay at infinity to transform the previous equation into the vector identity

$$\nabla(-\Delta)^{-\widetilde{s}}\phi_2 = -\beta_2\,y\,\phi_2^{2-\widetilde{m}}.$$

We pass now the term $\phi_2^{\widetilde{m}-1}$ to the left hand side, and finally, assuming regularity on ϕ_2 and taking divergence in both sides of the equation, we obtain

$$\nabla\cdot(\phi_2^{\widetilde{m}-1}\,\nabla(-\Delta)^{-\widetilde{s}}\phi_2) = -\beta_2\,\nabla\cdot(y\,\phi_2).$$

The regularity of ϕ_2 follows from the already proved regularity of ϕ_1 ([79]) and the correspondence (3.3.8). This is an a posteriori argument. In any case, without using the regularity, ϕ_2 is already a weak solution of problem (M2).

(2) The proof is similar to the first case. $\square$

Remarks. (i) Relation between the parameters

$$m \in [1,\infty) \longleftrightarrow \widetilde{m} \in [1,2)$$

$$m \in \left(\frac{N}{N+2s}, 1\right) \longleftrightarrow \widetilde{m} \in \left(\frac{N-2s}{N}, 1\right)$$

$$m \in \left(\frac{N-2s}{N}, \frac{N}{N+2s}\right) \longleftrightarrow \widetilde{m} \in \left(\frac{N-4s}{N-2s}, \frac{N-2s}{N}\right)$$

Notice that $m = 1$ implies $\widetilde{m} = 1$, which is the *Fractional Linear Heat Equation*. Since $\widetilde{m}_c < 1$, some singular cases of equation (M2) are covered where $\widetilde{m} < 1$. Thus, for $s = 1/2$ and $N = 2$ we get the whole range $\widetilde{m} \in (0, 2)$.

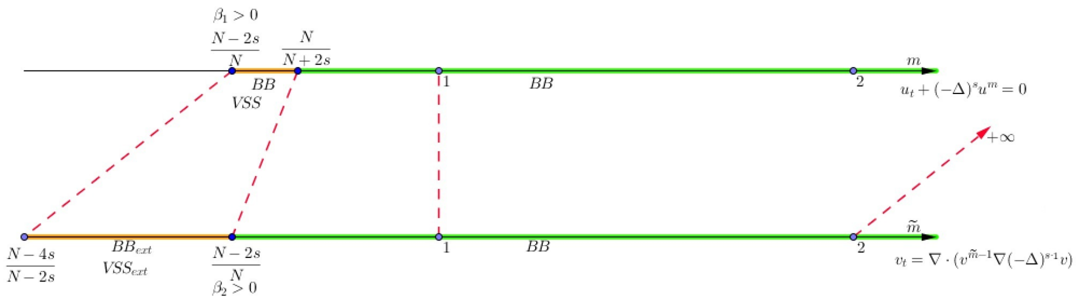

FIGURE 3.1: Related profiles of (M1) and (M2).

Figure 3.1 is drawn for N=2 and $s = \frac{1}{2}$. The notations stand for: BB=Barenblatt solution (type I), BB_{ext}=Barenblatt solution with extinction in finite time (type II), VSS=Very Singular Solution, VVS_{ext}=Very Singular Solution with extinction in finite time.

(ii) Conversely we can pass from a tripe $(\widetilde{s},\, \widetilde{m},\, \phi_2)$ for equation (M2) to the corresponding triple (s, m, ϕ_1) for equation (M1) through the relation

$$m = 1/(2 - \widetilde{m}), \quad s = 1 - \widetilde{s}, \quad \phi_1 = (\beta_2/\beta_1)^{\frac{1}{(1-m)}}\, \phi_2^{\frac{1}{m}}.$$

The following corollary describes the existence ranges and asymptotic behavior of the self-similar solutions of (M2). It comes as a consequence of our Theorem 3.3.1 and the previously known Theorem 3.2.2.

Corollary 3.3.2. *(i) For every $\widetilde{s} \in (0, 1)$ and $\widetilde{m} \in ((N - 2 + 2\widetilde{s})/N, 2)$ there is a fundamental solution of equation (M2) given by the formula (3.3.2). The behavior at infinity is given by*

$$\phi_2(x) \sim C|x|^{-(N+2-2\widetilde{s})/(2-\widetilde{m})}. \tag{3.3.12}$$

(ii) For every $\widetilde{s} \in (0, 1)$ and $m \in ((N - 4 + 4\widetilde{s})/(N - 2 + 2\widetilde{s}), (N - 2 + 2\widetilde{s})/N)$ there is a finite-time self-similar solution of type II given by the formula (3.3.4) with the asymptotic behavior

$$\psi_2(x) \sim C|x|^{-2(1-\widetilde{s})/(1-\widetilde{m})}. \tag{3.3.13}$$

(iii) For every $\widetilde{s} \in (0, 1)$ and $\widetilde{m} = (N - 2 + 2\widetilde{s})/N)$ there is a self-similar eternal in time of (M2) given by the formula (3.3.6). The behavior at infinity is given by

$$F(x) \sim |x|^{-N}. \tag{3.3.14}$$

In all cases the self-similar solutions have positive profiles. This is a partial confirmation that equation (M2) has infinite speed of propagation for all $\widetilde{m} \in (\widetilde{m}_*, 2)$, $\widetilde{m}_* = (N-4s)/(N-2s) < 1$. In the limit of this interval of infinite propagation we get the case $\widetilde{m} = 2$, i. e., the equation studied in [25] where finite propagation was established. Concerning general classes of solutions, we have proved infinite propagation in [69] for model (M2) for $\widetilde{m} \in (1,2)$ in dimension 1. Our corollary amounts to a partial result of infinite propagation in all dimensions for a range of $\widetilde{m}$ that goes below 1.

3.4 Model (M3). Equations with finite speed of propagation

Now we show a relation between the profile functions for equations when finite speed of propagation is expected. This happens in the third model (M3). This model has been studied by Biler, Karch and Monneau in [13] for $\hat{m} = 2$ describing dislocation phenomena in 1D, and for general $\hat{m}$ by Biler, Imbert and Karch in [12]. For non-negative initial data u_0 with suitable regularity properties there exists a unique weak solution $u(x,t)$ of Problem (M3). If $u_0 \geq 0$ then also $u(x,t) \geq 0$ for $t > 0$. A further characterization of the support of a general solution u is not known at this point, but particular solutions are found. In [12], the authors obtain a family of nonnegative explicit compactly supported self-similar solutions the model.

Let us examine de class of mass-preserving self-similar solutions. We observe that

$$w(x,t) = t^{-N\beta_3}\phi_2(y), \quad y = x\,t^{-\beta_3}$$

with

$$\beta_3 = 1/(N(\widehat{m}-1) + 2 - 2\widehat{s})$$

is a solution to equation (M3) if the profile ϕ_3 satisfies the equation

$$\nabla \cdot (\phi_3\,\nabla(-\Delta)^{-\widehat{s}}\phi_3^{\widehat{m}-1}) = -\beta_3 \nabla \cdot (y\,\phi_3) \tag{3.4.1}$$

and $w(x,t)$ has the property of mass conservation. Note that in the special case $\widehat{m} = 2$ the equation coincides with equation (M2) for $\widetilde{m} = 2$.

- **Case $\widetilde{m} = \widehat{m} = 2$.** In this particular case, the equation becomes

$$v_t = \nabla(v\nabla(-\Delta)^{-s}v). \tag{3.4.2}$$

The existence of a family of self-similar solutions of the Barenblatt type for equation (3.4.2) with compact support in the space variable has been proved independently by Caffarelli and Vázquez in [25] and by Biler, Imbert and Karch in [12]. The result proves that the profile $\phi_3 = \Phi$ is nonnegative, radially symmetric and compactly supported. In the latter reference, the authors obtain an explicit formula for the self-similar solution to equation (3.4.2) with

$$\Phi(y) = (a - b|y|^2)_+^{1-s}.$$

• **Case $\widehat{m} \geq 1$.** Paper [13] considers equation (M3) with $\widehat{m} > 1$ for which it presents self-similar solutions with the profile ϕ_3 given by the explicit formula

$$\phi_3(y) = \left(k(R^2 - |y|^2)_+^{1-s}\right)^{1/(m-1)}.$$

The regularity is Hölder continuous at the free boundary, $|x| = R\, t^{1/\beta_3}$.

Our present contribution in this instance is to show how this extension is also the result of a direct transformation of self-similarity profiles.

Theorem 3.4.1. *Let Φ be a solution to the profile equation (3.3.3) with $m = 2$, that is,*

$$\nabla \cdot (\Phi\, \nabla(-\Delta)^{-s}\Phi) = -\beta_2 \nabla \cdot (y\, \Phi), \qquad \beta_2 = (N + 2 - 2s)^{-1}. \tag{3.4.3}$$

Let $\widehat{m} > 1$. Let ϕ_3 defined by

$$\phi_3^{\widehat{m}-1} = (\beta_2/\beta_3)\, \Phi. \tag{3.4.4}$$

Then ϕ_3 is a solution to the profile equation

$$\nabla \cdot (\phi_3 \nabla(-\Delta)^{-\widehat{s}} \phi_3^{\widehat{m}-1}) = -\beta_3 \nabla \cdot (y\, \phi_3), \tag{3.4.5}$$

with $\widehat{s} = s$.

Proof. As in the proof of Theorem 3.3.1, from equation (3.4.3) we obtain the vector identity

$$\Phi \nabla(-\Delta)^{-s}\Phi = -\beta_2 y \Phi. \qquad y \in \mathbb{R}^N. \tag{3.4.6}$$

Let ϕ_3 given by (3.4.4), $\widehat{s} = s$ and $\widehat{m} > 1$ as given in the statement of the theorem. Then from (3.4.6) we obtain

$$\phi_3^{\widehat{m}-1}\, \nabla(-\Delta)^{-\widehat{s}}\, \phi_3^{\widehat{m}-1} = -\beta_3\, y\, \phi_3^{\widehat{m}-1},$$

and therefore

$$\phi_3\, \nabla(-\Delta)^{-\widehat{s}}\, \phi_3^{\widehat{m}-1} = -\beta_3\, y\, \phi_3.$$

We conclude that ϕ_3 is a solution the profile equation (3.4.5). $\square$

This theorem proves that self-similar solutions corresponding to parameters $\widehat{m} > 1$ are reduced to the computation of Φ, the profile function for $\widehat{m} = 2$. As a consequence of formula (3.4.4), it follows that for $\phi_3 = \phi_{3,\widehat{m}}$ the profile for $\widehat{m} > 1$, we have

$$\text{supp } \phi_3 = \text{supp } \Phi, \quad \text{for all } \widehat{m} > 1.$$

This means that the propagation of self-similar solutions does not depend on the parameter $\widehat{m}$.

3.5 A more general Fractional Porous Medium Equation

In order to see the previous transformations in a more general setting, we will study in this section the equation

$$z_t = \nabla(z^{\mathring{m}-1}\nabla(-\Delta)^{-\mathring{s}}z^{\mathring{n}-1}), \tag{MG}$$

which contains in particular the models (M2) and (M3) by including two different kinds of diffusion exponents. As far as we know, this model has not been studied before. Our goal is not to develop a complete theory of existence, uniqueness and regularity, but we want to show how the behavior of this equation is very related with the behavior of models (M2) and (M1).

More specifically, we consider self-similar solutions of the form

$$z(x,t) = t^{-N\beta_4}\phi_4(xt^{-\beta_4}) \tag{3.5.1}$$

where we have used conservation of mass as before. When considering self-similar solutions of the form (3.5.1), the two terms of equation (MG) in variable $y = xt^{-\beta_4}$ are as follows:

$$z_t(x,t) = -\beta_4\nabla_y \cdot (y\phi(y))t^{-N\beta_4-1},$$

$$\nabla_x \cdot (z^{\mathring{m}-1}\nabla_x(-\Delta)^{-\mathring{s}}z^{\mathring{n}-1}) = \nabla_y \cdot (\phi_4^{\mathring{m}-1}\nabla_y(-\Delta)^{-\mathring{s}}\phi_4^{\mathring{n}-1})t^{-\beta_4(N(\mathring{m}+\mathring{n}-2)+2-2s)}.$$

Let $\beta_4 = (N(\mathring{m}+\mathring{n}-3)+2-2\mathring{s})^{-1}$. The the profile ϕ_4 is a solution to the equation

$$\nabla \cdot (\phi_4^{\mathring{m}-1}\nabla(-\Delta)^{-\mathring{s}}\phi_4^{\mathring{n}-1}) = -\beta_4\nabla \cdot (y\phi_4). \tag{3.5.2}$$

In the following theorem , we will prove that the self-similar solutions of this class are fully characterized when $\mathring{n} > 1$ by the self-similar solutions of models (M1) and (M2) presented in the previous sections.

Theorem 3.5.1. *Let ϕ_4 be the solution of the profile equation (3.5.2)*

$$\nabla \cdot (\phi_4^{\mathring{m}-1} \nabla (-\Delta)^{-\mathring{s}} \phi_4^{\mathring{n}-1}) = -\beta_4 \nabla \cdot (y\phi_4).$$

1. *Let $\mathring{n} > 1$,$\mathring{m} < 2$ and ϕ_1 the solution to*

$$(-\Delta)^s \phi_1^m = \beta_1 \nabla \cdot (y\phi_1),$$

for $m = \frac{\mathring{n}-1}{2-\mathring{m}}$ and $s = 1 - \mathring{s}$. Then $\phi_1 = \left(\frac{\beta_1}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{2-\mathring{m}}$.

2. *Let $\mathring{n} > 1$,$\mathring{m} \geq 2$ and ϕ_2 the solution to*

$$\nabla \cdot (\phi_2^{\widetilde{m}-1} \nabla (-\Delta)^{-\widetilde{s}} \phi_2) = -\beta_2 \nabla \cdot (y\phi_2),$$

for $\widetilde{m} = \frac{2\mathring{n}+\mathring{m}-4}{\mathring{n}-1}$. and $\widetilde{s} = \mathring{s}$. Then $\phi_2 = \left(\frac{\beta_2}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{\mathring{n}-1}$.

Proof. As before, we consider the vectorial expression that can be directly deduced from (3.5.2)

$$\nabla(-\Delta)^{-\mathring{s}} \phi_4^{\mathring{n}-1} = -\beta_4 y \phi_4^{2-\mathring{m}}. \tag{3.5.3}$$

Consider also the corresponding vectorial profile equation

$$\nabla(-\Delta)^{s-1} \phi_1^m = \beta_1 y \phi_1.$$

Let $\mathring{n} > 1$ and $\mathring{m} < 2$. The profile $\phi_1 = \left(\frac{\beta_1}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{2-\mathring{m}}$ for $m = \frac{\mathring{n}-1}{2-\mathring{m}}$ and $s = 1 - \mathring{s}$ transform the last profile equation in to (3.5.3).

In the same way, for $\mathring{n} > 1$ and $\mathring{m} \geq 2$ the vectorial profile

$$\nabla(-\Delta)^{-\widetilde{s}} \phi_2 = -\beta_2 y \phi_2^{2-\widetilde{m}}$$

turns into (3.5.3) if $\phi_2 = \left(\frac{\beta_2}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{\mathring{n}-1}$ for $\widetilde{m} = \frac{2\mathring{n}+\mathring{m}-4}{\mathring{n}-1}$ and $\widetilde{s} = \mathring{s}$. We summarize the results of this theorem in Figure 3.2. □

We summarize the results on (MG) in Figure 3.2 as follows. The horizontal axis is the $\mathring{m}$ variable, the vertical axis is the $\mathring{n}$ variable. (a) The $\mathring{m} < 2$ left part describes the connection between (MG) and (M1) according to Theorem 3.5.1-1: each green line is defined by $\mathring{n} = m(2 - \mathring{m}) + 1$ for particular values of m that are mentioned at the lower

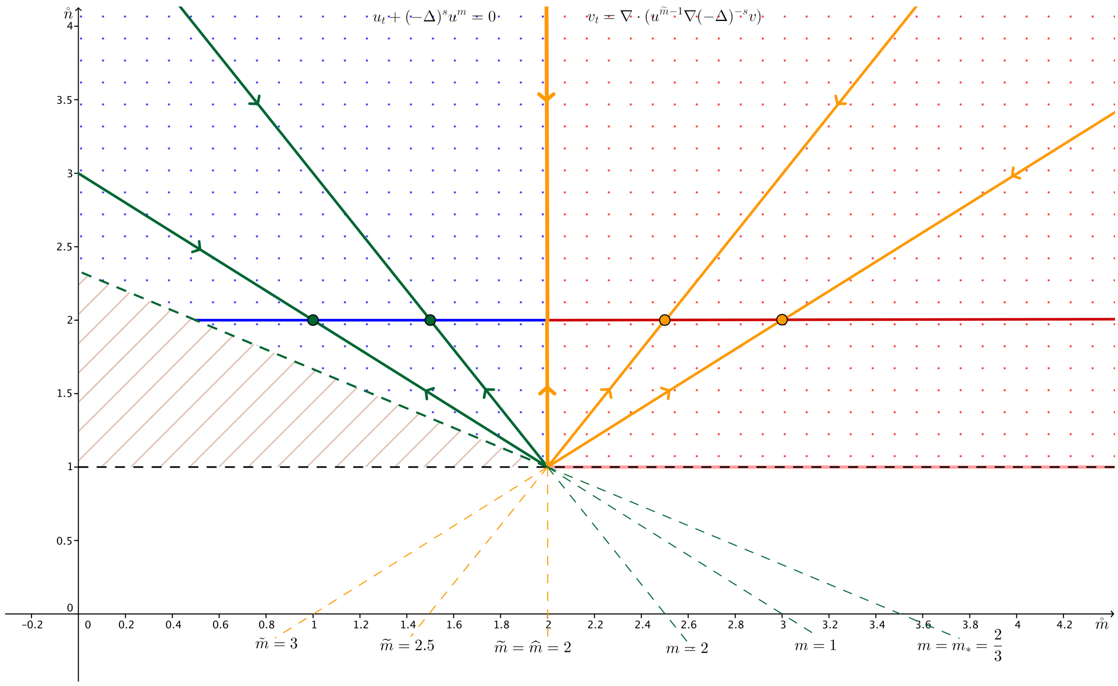

FIGURE 3.2: Related profiles of (MG) with (M1) and (M2) for N=3 and $s = \frac{1}{2}$.

extremity of the line. The profile function ϕ_4 of (MG)-$(\mathring{m}, \mathring{n})$ is obtained from the profile ϕ_1 of (M1)-(m) for every $(\mathring{m}, \mathring{n})$ on the corresponding green line. As a consequence, ϕ_4 for every $(\mathring{m}, \mathring{n})$ on the green line can be obtained from ϕ_4 for $(2 - 1/m, 2)$, the point on the line with $\mathring{n} = 2$. This reduction to the case $\mathring{n} = 2$ is suggested by the arrows converging to the point $(2 - 1/m, 2)$. (b) Similarly the $\mathring{m} \geq 2$ right part describes the connection between (MG) and (M2) according to Theorem 3.5.1-2. Each orange line is defined by $\mathring{n}(\mathring{m} - 2) = \mathring{m} - 4 + \widetilde{m}$ for particular values of $\widetilde{m}$. The profile ϕ_4 of (MG)$-(\mathring{m}, \mathring{n})$ is obtained in this case from the ϕ_2 of (M2)$-(\widetilde{m})$. As a consequence, ϕ_4 of (MG)$-(\mathring{m}, \mathring{n})$ is reduced to ϕ_4 of (MG)-$(\widetilde{m}, 2)$ which is the point corresponding to $\mathring{n} = 2$ the the respective orange line. (c) The vertical line $\mathring{m} = 2$ corresponds (M3) studied in [12]. The horizontal line $\mathring{n} = 2$ corresponds to (M2) studied in [69]: blue for infinite propagation, while red is for finite propagation.

Note 3.5.1. We should note the $N = 3$ and $s = \frac{1}{2}$ plays no special role in the figure. The only determinate the shape of the line correspondent to $m = m_* = \frac{N-2s}{N} = \frac{2}{3}$.

Note 3.5.2. Notice that the relation $\phi_1 = \left(\frac{\beta_1}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{2-\mathring{m}}$ for $\mathring{n} > 1$ and $\mathring{m} < 2$ is also true in some sense if $\mathring{n} < 1$ and $\mathring{m} \geq 2$. It is not very clear the meaning of model (MG) for $\mathring{n} < 1$, but this relation strongly changes the behavior of the profile equation. For example, decay at infinity does not hold anymore.

The same happens for $\phi_2 = \left(\frac{\beta_2}{\beta_4}\right)^{\frac{1}{m-1}} \phi_4^{\mathring{n}-1}$ if we are in the range $\mathring{n} < 1$ and $\mathring{m} < 1$.

3.6 Very singular solutions for (M2)

In this section we investigate another important class of solutions for the equation (M2) in a certain range of parameters $\widetilde{m}$ corresponding to Fractional Fast Diffusion Equations. These solutions are called Very Singular Solutions (VSS).

3.6.1 Very singular solutions of type I for (M2)

Solutions of this type are obtained in [79] for model M1 by the method of separation of variables. The same argument could be done here for (M2) , but instead of it, we will make use of the profile equation (3.3.3) that is very familiar for us at this point. We will look for solutions of the form $\phi(y) = C|y|^{-\alpha}$ to the corresponding profile equation. We recall that the profile equation for model (M2) is

$$\nabla \cdot (\phi^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}\phi) = -\beta_2 \nabla \cdot (y\phi). \tag{3.3.3}$$

In the sequel we will denote $y = y_i$ to clarify the vectorial notation required. The profile equation implies that

$$\phi^{\widetilde{m}-1}\frac{\partial}{\partial y_i}(-\Delta)^{-\widetilde{s}}\phi = -\beta_2 y_i \phi.$$

In Appendix 1, we review the fact that $(-\Delta)^{-\widetilde{s}}(|y|^{-\alpha}) = k(\alpha)|y|^{-\alpha+2\widetilde{s}}$ where $k(\alpha)$ is an explicit constant. In this way,

$$C^m k(\alpha)|y|^{-\alpha(\widetilde{m}-1)}(-\alpha+2\widetilde{s})y_i|y|^{-\alpha+2\widetilde{s}-2} = -C\beta_2 y_i|y|^{-\alpha},$$

which gives the following simplified equation for the solution

$$C^{m-1}|y|^{-\alpha\widetilde{m}+2\widetilde{s}-2} = -\frac{\beta_2}{k(\alpha)(-\alpha+2\widetilde{s})}|y|^{-\alpha}.$$

Therefore, ϕ is a solution to equation (3.3.3) if the following equalities hold true

$$-\alpha\widetilde{m}+2\widetilde{s}-2 = -\alpha, \quad C^{\widetilde{m}-1} = -\frac{\beta_2}{k(\alpha)(-\alpha+2\widetilde{s})}.$$

These conditions determine the exact values of the coefficient C and the exponent α:

$$\alpha = \frac{2-2\widetilde{s}}{1-\widetilde{m}} \qquad \text{and} \qquad C^{1-\widetilde{m}} = 2\overline{k}(\alpha)\frac{1-\widetilde{s}(2-\widetilde{m})}{(1-\widetilde{m})\beta_2}.$$

where $k(\alpha)$ is calculated explicitly in appendix 1. In this way, the condition of existence of VSS for model (M2) is the existence of such a constant C. We need to show the range of $\widetilde{m}$ that ensures

$$2^{1-2s}\frac{\Gamma\left(\frac{N-\alpha}{2}\right)\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha+2\widetilde{s}}{2}\right)}\cdot\frac{1-\widetilde{s}(2-\widetilde{m})}{(1-\widetilde{m})\beta_2}>0. \tag{3.6.1}$$

The following to properties of the Γ-functions are used

$$\Gamma(x)>0 \text{ if } x>0. \qquad \Gamma(x)<0 \text{ if } x\in(-1,0).$$

The first condition is $\widetilde{m}<1$ to make $\alpha>0$. In this way, $\Gamma(\alpha/2)>0$. We also need $\beta_2>0$ to ensure the decay at infinity of the VSS. This restriction implies the first condition on $\widetilde{m}$,

$$\widetilde{m}>\frac{N-2+2\widetilde{s}}{N}. \tag{3.6.2}$$

We observe that for

$$\widetilde{m}>\frac{N+2\widetilde{s}}{N+2} \tag{3.6.3}$$

we get $\frac{N-\alpha}{2}=-\frac{1}{2(1-\widetilde{m})\beta_2}>-1$ and so on $\Gamma\left(\frac{N-\alpha}{2}\right)\frac{1}{\beta_2}<0$. It is also easy to check that $\frac{\alpha-2\widetilde{s}}{2}=\frac{1-\widetilde{s}(2-\widetilde{m})}{(1-\widetilde{m})}>-1$, which implies

$$\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}\right)\frac{1-\widetilde{s}(2-\widetilde{m})}{(1-\widetilde{m})}>0.$$

At this point, we must have $\Gamma\left(\frac{N-\alpha+2\widetilde{s}}{2}\right)<0$ to make (3.6.2) hold. Se we need

$$-1<\frac{N-\alpha+2\widetilde{s}}{2}<0\,,$$

which gives the following two conditions on $\widetilde{m}$:

$$\widetilde{m}>\frac{N-2+4\widetilde{s}}{N+2\widetilde{s}}, \qquad \widetilde{m}<\frac{N+4\widetilde{s}}{N+2+2\widetilde{s}}. \tag{3.6.4}$$

We get the existence result by choosing the more restrictive conditions between (3.6.2), (3.6.3) and (3.6.4).

Theorem 3.6.1. *There exists a Very Singular Solution to model* (M2) *of the form*

$$v(x,t)=Kt^{-\frac{1}{1-\widetilde{m}}}|x|^{-\frac{2-2\widetilde{s}}{1-\widetilde{m}}}$$

for all $\widetilde{m}$ *in the interval* $\left(\frac{N-2+4\widetilde{s}}{N+2\widetilde{s}},\frac{N+2\widetilde{s}}{N+2}\right)$.

Proof. We have obtained a solution $v(x,t)=t^{-N\beta_2}\phi(xt^{-\beta_2})$ for $\phi(y)=C|y|^{-\frac{2-2\widetilde{s}}{1-\widetilde{m}}}$ and exponent $\beta_2=(N(\widetilde{m}-1)+2-2\widetilde{s})^{-1}$. The form of the solution v is as follows

$$v(x,t)=Kt^{\beta_2\left(\frac{2-2\widetilde{s}}{1-\widetilde{m}}-N\right)}|x|^{-\frac{2-2\widetilde{s}}{1-\widetilde{m}}},$$

with a suitable constant $K = K(N, \widetilde{s}, \widetilde{m})$. Since $\beta_2 \left(\frac{2-2\widetilde{s}}{1-\widetilde{m}} - N\right) = -\frac{1}{1-\widetilde{m}}$, then $v(x,t)$ can be written in a simpler form:

$$v(x,t) = Kt^{-\frac{1}{1-\widetilde{m}}}|x|^{-\frac{2-2\widetilde{s}}{1-\widetilde{m}}}. \tag{3.6.5}$$

□

Note that these VSS are algebraically the same that the ones obtained by separation of variables in ([79]) for equation (M1) if $\widetilde{m} = m$ and $\widetilde{s} = 1 - s$, except for the constant K.

3.6.2 Very singular solutions with extinction for (M2)

We search for solutions of the form

$$U(x,t) = B(t)|x|^{-\alpha}$$

of equation (M2). They have to satisfy the relation

$$B'(t)|x|^{-\alpha} = B(t)^m \mathcal{C}|x|^{-\alpha\widetilde{m}+2\widetilde{s}-2}$$

where

$$\mathcal{C} = \overline{k}(\alpha)(-\alpha + 2\widetilde{s})(-\alpha\widetilde{m} + 2\widetilde{s} - 2 + N)$$

(see the computations in the previous case). Therefore

$$\alpha = \frac{2-2s}{1-m}$$

and

$$B'(t) = \mathcal{C}B(t)^m. \tag{3.6.6}$$

We study now the sign of the constant $\mathcal{C}$:

$$\mathcal{C} = 2^{1-2s} \frac{\Gamma\left(\frac{N-\alpha}{2}\right)\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha+2\widetilde{s}}{2}\right)} \cdot \frac{-\alpha+2\widetilde{s}}{2} \cdot \frac{-1}{\beta_2(1-m)},$$

where $\beta_2^{-1} = N(m-1) + 2 - 2\widetilde{s}$.

We consider now the case $\beta_2 < 0$ that is $\widetilde{m} < \frac{N-2+2\widetilde{s}}{N}$ and then we obtain $\alpha < N$. Therefore all of Gamma functions above are positive, but for $\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}\right)$ whose sign we do not estimate. We observe that

$$\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}\right)\cdot\frac{-\alpha+2\widetilde{s}}{2}=-\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}+1\right).$$

Since $\frac{\alpha-2\widetilde{s}}{2}+1=(\alpha+2-2\widetilde{s})/2>0$ then $\Gamma\left(\frac{\alpha-2\widetilde{s}}{2}+1\right)>0$. We obtain therefore that $\mathcal{C}<0$. Then, solving equation (3.6.6) with $B(T)=0$, we obtain the following formula for $B(t)$:

$$B(t)=\left(-\mathcal{C}(1-m)\right)^{1/(1-m)}(T-t)^{1/(1-m)}.$$

Theorem 3.6.2. *Let $\widetilde{s}\in(0,1)$, $0<\widetilde{m}<\frac{N-2+2\widetilde{s}}{N}$. Then there exists a Very Singular Solution to model (M2) of the form*

$$v(x,t)=K(T-t)^{1/(1-m)}|x|^{-\frac{2-2\widetilde{s}}{1-\widetilde{m}}}\quad \textit{for } t\in[0,T]$$

and $v(x,t)=0$ for $t>T$.

Note that these VSS are algebraically the same that the ones obtained by Vázquez and Volzone for (M1) in [84] if $\widetilde{m}=m$ and $\widetilde{s}=1-s$, except for the constant K.

3.7 Appendix 1: Inverse fractional Laplacians and Potentials

The definition of $(-\Delta)^w$ is also done by means of Fourier transform

$$\mathcal{F}((-\Delta)^s f)(\xi)=(2\pi|\xi|)^{2s}\mathcal{F}(f)(\xi),$$

and can be use even for negative values of s. In the range $N/2<s<0$ we have an equivalent definition in terms of a Riesz potential

$$(-\Delta)^{-s}f(x)=I_s(f)=\gamma(s)^{-1}\int_{\mathbb{R}^N}\frac{f(y)}{|x-y|^{N-2s}}dy,$$

acting on functions of the class $\mathcal{S}$. The function γ is defined by

$$\gamma(\rho)=\pi^{N/2}\,2^{\rho}\frac{\Gamma(\rho/2)}{\Gamma((N-\rho)/2)}.$$

Note that $\gamma(\rho)\to\infty$ as $\rho\to N$, but $\gamma(s)/(N-\rho)$ converge to the nonzero constant $\pi^{N/2}2^{N-1}\Gamma(N/2)$.

It is well known that the Fourier Transform of the function $f_\alpha(x) = |x|^{-\alpha}$ is

$$\mathcal{F}(f_\alpha)(\xi) = \gamma(N-\alpha)(2\pi)^{\alpha-N}|\xi|^{\alpha-N}.$$

In this way, we can compute $(-\Delta)^{-s} f_\alpha(x)$ as follows,

$$\begin{aligned} \mathcal{F}((-\Delta)^{-s} f_\alpha)(\xi) &= (2\pi|\xi|)^{-2s}\mathcal{F}(f_\alpha)(\xi) = \gamma(N-\alpha)(2\pi)^{\alpha-N-2s}|\xi|^{\alpha-N-2s} \\ &= \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}\gamma(N-\alpha+2s)(2\pi)^{\alpha-N-2s}|\xi|^{\alpha-N-2s} \\ &= \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}\mathcal{F}(f_{\alpha-2s})(\xi), \end{aligned}$$

that is

$$(-\Delta)^{-s} f_\alpha(x) = \overline{k}(\alpha) f_{\alpha-2s}(x), \qquad \overline{k}(\alpha) = \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}.$$

More exactly

$$\overline{k}(\alpha) = 2^{-2s}\frac{\Gamma((N-\alpha)/2)\,\Gamma\left((\alpha-2s)/2\right)}{\Gamma(\alpha/2)\,\Gamma((N-\alpha+2s)/2)}.$$

3.8 Appendix 2: Comments and open problems

The following questions appear naturally in view of the results of this chapter.

• To decide if the asymptotic behavior of a general solution of (M2) and (M3) is given by a Barenblatt type solution. This fact is well known for (M1) for general $m > (N-2s)_+/N$ (see [79]) and for (M2), (M3) with $\widetilde{m} = 2$, $\mathring{m} = 2$ (see [24]).

• To find explicit or semi-explicit formulas for the Barenblatt profiles of models (M1) and (M2).

• To find explicit or semi-explicit solutions of any kind for model (M2) with $\widetilde{m} > 2$.

• Is it possible to find a transformation between general solutions of (M1), (M2), (M3) and (MG)?

• Develop a general theory for Model (MG).

• Develop numerical methods for models (M2), (M3) and (MG).

Chapter 4

Uniqueness and properties of distributional solutions of nonlocal degenerate diffusion equations of porous medium type

4.1 Introduction

In this chapter, we obtain uniqueness, existence, and various other properties for bounded distributional solutions of a class of possibly degenerate nonlinear anomalous diffusion equations of the form:

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \qquad \text{in} \quad Q_T := \mathbb{R}^N \times (0,T) \tag{4.1.1}$$

$$u(x,0) = u_0(x) \qquad \text{on} \quad \mathbb{R}^N \tag{4.1.2}$$

where $u = u(x,t)$ is the solution and $T > 0$. The nonlinearity φ is an arbitrary continuous nondecreasing function, while the anomalous or nonlocal diffusion operator $\mathcal{L}^\mu$ is defined for any $\psi \in C_c^\infty(\mathbb{R}^N)$ as

$$\mathcal{L}^\mu[\psi](x) = \int_{\mathbb{R}^N\setminus\{0\}} \psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1} \,\mathrm{d}\mu(z), \tag{4.1.3}$$

where D is the gradient, $\mathbf{1}_{|z|\leq 1}$ a characteristic function, and μ a nonnegative symmetric possibly singular measure satisfying the Lévy condition $\int |z|^2 \wedge 1 \,\mathrm{d}\mu(z) < \infty$. For the precise assumptions, we refer to Section 4.2.

The class of nonlocal diffusion operators we consider coincide with the generators of the symmetric pure-jump Lévy processes [7] like e.g. compound Poisson processes, CGMY processes in Finance, and symmetric s-stable processes. Included are the well-known fractional Laplacians $-(-\Delta)^{\frac{s}{2}}$ for $s \in (0,2)$ (where $\mathrm{d}\mu(z) = c_s \frac{\mathrm{d}z}{|z|^{N+s}}$ for some $c_s > 0$ [7, 43]), along with degenerate operators, and surprisingly, numerical discretizations of these operators!

In the language of [77], equation (4.1.1) is a generalized porous medium equation. Since φ is only assumed to be continuous, the full range of porous medium nonlinearities is included: $\varphi(r) = r|r|^{m-1}$ for $m > 0$. In particular, also the fast diffusion range. On the other hand, since φ is only assumed to be nondecreasing, it can be constant on sets of positive measure and then equation (4.1.1) is strongly degenerate. This case include Stefan type of problems, like e.g. when $c_1, c_2, T > 0$ and

$$\varphi(r) = \begin{cases} c_2 r, & r < 0, \\ c_1 (r-T)^+, & r \geq 0. \end{cases}$$

Many physical problems can be modelled by equations like (4.1.1). We mention flow in a porous medium of e.g. oil, gas, and groundwater, nonlinear heat transfer, and population dynamics. For more information and examples, we refer to chapter 2 and 21 in [77] for local problems, and to [7, 65, 81, 85] for nonlocal problems.

A key result in this chapter is the uniqueness result for bounded distributional solutions of (4.1.1) and (4.1.2). Almost half of the chapter is devoted to the proof of this result. Once we have it, we prove a general stability result, and then we obtain other properties like existence, L^1-contraction, and many a priori estimates from more regular problems via approximation and compactness arguments. As straightforward applications of all of these estimates, we then obtain the following results: (i) Convergence as $s \to 2^-$ of distributional solutions of

$$\partial_t u + (-\Delta)^{\frac{s}{2}} \varphi(u) = 0 \qquad \text{in} \qquad Q_T, \tag{4.1.4}$$

to distributional solutions of the local equation

$$\partial_t u - \Delta \varphi(u) = 0 \qquad \text{in} \qquad Q_T; \tag{4.1.5}$$

(ii) continuous dependence in $(m,s) \in (0,\infty) \times (0,2]$ for the porous medium equation of [38],

$$\partial_t u + (-\Delta)^{\frac{s}{2}} u|u|^{m-1} = 0 \qquad \text{in} \qquad Q_T, \tag{4.1.6}$$

including for the first time also the fast diffusion range; and (iii) convergence of semi-discrete numerical approximations of a class of equations including (4.1.1) (cf. (4.2.7) and (4.2.8) in Section 4.2.2).

The uniqueness result is hard to prove because of our very general assumptions on the initial value problem combined with a very weak solution concept—merely bounded distributional solutions. This combination means that many classical techniques do not work: Fourier techniques are hard to apply because the problem is nonlinear and the Fourier symbol of $\mathcal{L}^\mu$ could be merely a bounded function, energy estimates do not imply uniqueness because φ is not strictly increasing, and L^1-contraction arguments do not apply since we do not assume additional entropy conditions (cf. e.g. [5] for the local case), or equivalently, additional regularity in time as in [38] (see the uniqueness result for so-called strong solutions). Finally, since our solutions are not assumed to have finite energy, the classical uniqueness argument of Oleinik [66] cannot be adapted either. For this type of results, we refer to [66, 77] for the local case, and the uniqueness argument for so-called weak solutions in [38, 48] for results in the nonlocal case.

For the local equation (4.1.5), uniqueness for bounded distributional solution was proved by Brezis and Crandall in [19] under similar assumptions on φ and u_0. Their argument is quite indirect and rely on a clever idea using resolvents and their integral representations (fundamental solutions). In this chapter, we adapt such an approach to our nonlocal setting. But because of the generality of our diffusion operators, we cannot rely on explicit fundamental solutions for our proofs. In stead, we have to develop this part of the theory from scratch, using the equation and the regularity that comes with our solutions concept to obtain the necessary estimates. To do this, a key tool is to approximate the possibly singular integral operator $\mathcal{L}^\mu$ by a bounded integral operator and then carefully pass to the limit. This proceedure, and hence also the proof, is truly nonlocal—there is no similar approximation by local operators. The proof necessarily becomes much more involved than in [19], and includes a number of approximations, a priori estimates, L^1-contraction estimates, comparison principles, compactness and regularity arguments. It also includes new Stroock-Varoupolous inequalities and a new Liouville type of result for nonlocal operators. Both our approach and intermediate results should be of independent interest.

Let us give the main references for the well-posedness of the Cauchy problems for (4.1.1) and (4.1.5). We start with the local case (4.1.5). In the linear case, when $\varphi(u) = u$, it is the classical heat equation, cf. e.g. [47]. When $\varphi(u) = u^m$, it is a porous medium equation, and a very complete theory can be found in [77]. In the general case, (4.1.5) is a generalized porous medium equation (or filtration equation). We refer again to [77]. Uniqueness of distributional solutions of this equation was proved in [19] for bounded initial data and continuous, nondecreasing φ and in [50] for locally integrable initial data, $\varphi(r) = r^m$ with $0 < m < 1$, and with regularity assumptions on $\partial_t u$. In the presence of convection, or if general L^1-contraction results are sought, then so-called entropy solutions are a useful tool to obtain well-posedness [26, 62]. A very general well-posedness result which cover the case of merely continuous φ can then be found in [5].

In the nonlocal case, one linear special case of (4.1.1) is the popular fractional heat equation $\partial_t u + (-\Delta)^{\frac{s}{2}} u = 0$ for $s \in (0, 2)$. As in the local case, the unique classical solution is given by $u(x,t) = (K_s(\cdot,t) * u(\cdot,0))(x)$ for $\mathcal{F}(K_s(\cdot,t))(\xi) = \mathrm{e}^{-|\xi|^s t}$. The fractional porous medium equations (4.1.6) are examples of nonlinear equations of the form (4.1.1). In [37, 38] existence, uniqueness and a priori estimates for (4.1.6) are proved for a so-called weak L^1-energy solutions—possibly unbounded solutions with finite energy. We also mention that logarithmic diffusion ($\varphi(u) = \log(1+u)$) is considered in [39], and that problems on bounded domains have been studied in [16, 18]. Another class of equations on the form (4.1.1), are many of the equations of [6]. These equations involve bounded diffusion operators that can be represented by nonsingular integral operators of the form (4.1.3). Because of this, at least the well-posedness is easier to handle in this case.

There are other ways to generalize the porous medium equation to a nonlocal setting. In [12, 25, 69, 70] the authors consider a so-called porous medium equations with fractional pressure. These equations are in a divergence form, and no uniqueness is known except when $N = 1$. Finally, we mention that in the presence of (nonlinear) convection, additional entropy conditions are needed to have uniqueness as in the local case. Nonuniqueness of distributional solutions is proved in [2], and several well-posedness results for entropy solutions are given in [1, 31, 45]. These latter results requires φ to linear or locally Lipschitz and hence does not apply to our case where φ is merely continuous.

4.1.1 Outline

In Section 4.2 we state the assumptions and present and discuss our main results. The proof of the uniqueness results is given in Section 4.3. This proof requires a number of results and estimates for a resolvent equation—an auxiliary elliptic equation—and these are proved in Section 4.6. In Section 4.4, we prove the main stability and existence result, along with a number of a priori estimates. We then apply these results to prove the convergence to the local case, continuous dependence, and the properties and convergence of the numerical scheme in Section 4.5. Finally, after Section 4.6, there is an appendix with the proofs of some technical results.

4.1.2 Notation

For $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$, and $\operatorname{sign}^+(x)$ is $+1$ for $x > 0$ and 0 for $x \leq 0$. We let $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, $\mathbf{1}_A(x)$ be 1 for $x \in A \subset \mathbb{R}^N$ and 0 otherwise, and $\operatorname{supp}\psi$ be the support of a function ψ. Derivatives are denoted by $'$, $\frac{d}{dt}$, ∂_t, ∂_{x_i}, and $D\psi$ and $D^2\psi$ denote the x-gradient and Hessian matrix of ψ. Convolution is defined as $f * g(x) = [f*g](x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,\mathrm{d}y$, and $(f,g) = \int_{\mathbb{R}^N} fg\,\mathrm{d}x$ whenever the integral is well-defined. If $f, g \in L^2(\mathbb{R}^N)$, we write $(f,g)_{L^2(\mathbb{R}^N)}$. The L^2-adjoint of an operator $\mathcal{T}$ is denoted by $\mathcal{T}^*$, and the reader may check that $(\mathcal{L}^\mu)^* = \mathcal{L}^{\mu^*}$ (see below for the definition of μ^*). A modulus of continuity is a nonnegative function $\lambda(\varepsilon)$ which is continuous in ε with $\lambda(0) = 0$. By a classical solution, we mean a solution such that the equation holds pointwise everywhere.

Function spaces: C_0, C_{b}, C_{b}^∞ and C_{c}^∞ are spaces of continuous functions that are vanishing at infinity; bounded; bounded with bounded derivatives of all orders; and smooth functions with compact support respectively. $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is the space of measurable functions $\psi : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that (i) $\psi(\cdot, t) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for every $t \in [0,T]$; (ii) for all compact $K \subset \mathbb{R}^N$, $\int_K |\psi(x,t) - \psi(x,s)|\,\mathrm{d}x \to 0$ when $t \to s \in [0,T]$; and (iii) $\|\psi\|_{C([0,T];L^1(K))} := \operatorname{ess\,sup}_{t\in[0,T]} \int_K |\psi(x,t)|\,\mathrm{d}x < \infty$.

Measures: $\delta_a(x)$ denotes the delta measure centered at $a \in \mathbb{R}^N$. Let $X \subset \mathbb{R}^N$ be open and μ a Borel measure on X. For $x \in X$ and $\Omega \subset X$ Borel, we denote $\mu_x(\Omega) = \mu(\Omega + x)$ where $\Omega + x = \{y + x : y \in \Omega\}$. Moreover, μ^* is defined as $\mu^*(B) = \mu(-B)$ for all Borel sets B, and we say that μ is symmetric if $\mu^* = \mu$. The support of a Borel measure μ on is

$$supp\mu = \{x \in X : \mu(B(x,r) \cap X) > 0 \text{ for all } r > 0\}.$$

The Lebesgue measure of $\mathbb{R}^N$ is denoted by $\mathrm{d}w$ if w is a generic variable on $\mathbb{R}^N$. Moreover, the tensor product $\mathrm{d}\mu(z)\,\mathrm{d}w$ is a well-defined nonnegative Radon measure since μ is σ-finite (for more details, consult [3, Section 2.1.2].)

For the rest of the chapter, we fix two families of mollifiers ω_δ, ρ_δ defined by

$$\omega_\delta(\sigma) := \frac{1}{\delta^N}\omega\left(\frac{\sigma}{\delta}\right) \tag{4.1.7}$$

for fixed $0 \leq \omega \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ satisfying $\operatorname{supp}\omega \subseteq \overline{B}(0,1)$, $\omega(\sigma) = \omega(-\sigma)$, $\int \omega = 1$; and

$$\rho_\delta(\tau) := \frac{1}{\delta}\rho\left(\frac{\tau}{\delta}\right) \tag{4.1.8}$$

for fixed $0 \leq \rho \in C_c^\infty([0,T])$, $\operatorname{supp}\rho \subseteq [-1,1]$, $\rho(\tau) = \rho(-\tau)$, $\int \rho = 1$.

4.2 The main results

In this section, we present the main results: first of all uniqueness, and then stability, existence and a number of estimates for the solutions of (4.1.1) and (4.1.2). As an application of our main results, we give compactness and continuous dependence estimates. We introduce a semi-discrete numerical scheme for even more general equations and show that convergence and other properties easily follow from our previous results. Finally, we establish a new existence result that also cover local diffusion equations.

Throughout the chapter we assume that

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing;} \tag{A_φ}$$

$$u_0 \in L^\infty(\mathbb{R}^N); \tag{A_{u_0}}$$

$$\mu \text{ is a nonnegative symmetric Radon measure on } \mathbb{R}^N \setminus \{0\} \text{ satisfying} \tag{A_μ}$$
$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

Remark 4.2.1. (a) Without loss of generality, we can assume $\varphi(0) = 0$ (by adding a constant to φ).

(b) A nonlocal operator defined by (4.1.3) is a nonpositive operator (see Lemma 4.3.7).

We use the following definition of distributional solutions of (4.1.1) and (4.1.2).

Definition 4.2.2. Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $u \in L^1_{\mathrm{loc}}(Q_T)$. Then

(a) u is a distributional solution of equation (4.1.1) if

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \quad \text{in} \quad \mathcal{D}'(Q_T),$$

(b) u is a distributional solution of the initial condition (4.1.2) if

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} u(x,t)\psi(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0,T)).$$

The equation in part (a) is well defined when e.g. (A_φ) and (A_μ) hold and $u \in L^\infty(Q_T)$. Note as well that the initial condition u_0 is assumed in the distributional sense (u_0 is a weak initial trace). See Lemma 4.2.20 below for an equivalent definition.

We state the main result of this chapter.

Theorem 4.2.3. *Assume* (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T), \tag{4.2.1}$$

$$u - \hat{u} \in L^1(Q_T), \tag{4.2.2}$$

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \qquad \textit{in} \qquad \mathcal{D}'(Q_T) \tag{4.2.3}$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \textit{for all} \quad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T)). \tag{4.2.4}$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Sections 4.3 and 4.6 are devoted to the (long) proof of this result.

Corollary 4.2.4 (Uniqueness). *Assume* (A_φ), (A_{u_0}) *and* (A_μ). *Then there is at most one distributional solution* u *of* (4.1.1) *and* (4.1.2) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof. Assume there are two solutions u and $\hat{u}$. Then all assumptions of Theorem 4.2.3 obviously hold ($\|u-\hat{u}\|_{L^1} \leq \|u-u_0\|_{L^1} + \|\hat{u}-u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. □

Remark 4.2.5. Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0 are not included. In Section 4.2.3 below we discus some extensions of the uniqueness result.

Next, we study under which assumptions solutions of

$$\partial_t u_n - \mathcal{L}^{\mu_n}[\varphi_n(u_n)] = 0 \quad \text{in} \quad Q_T, \tag{4.2.5}$$

converge to solutions of

$$\partial_t u - \mathcal{L}[\varphi(u)] = 0 \quad \text{in} \quad Q_T. \tag{4.2.6}$$

Theorem 4.2.6 (Stability). *Assume* $\mathcal{L} : C_\mathrm{c}^\infty(Q_T) \to L^1(Q_T)$, μ_n *satisfies* (A_μ), φ_n *and* φ *satisfy* (A_φ), *and* $u_n, u \in L^\infty(Q_T)$ *for every* $n \in \mathbb{N}$. *Then if* $\{u_n\}_{n\in\mathbb{N}}$ *is a sequence of distributional solutions of* (4.2.5), $\sup_n \|u_n\|_{L^\infty(Q_T)} < \infty$, *and*

(i) $\mathcal{L}^{\mu_n}[\psi] \to \mathcal{L}[\psi]$ *in* $L^1(\mathbb{R}^N)$ *for all* $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$;

(ii) $\varphi_n \to \varphi$ *locally uniformly;*

(iii) $u_n \to u$ *pointwise a.e. in* Q_T;

then u *is a distributional solution of* (4.2.6).

This result is proved in Section 4.4.

Remark 4.2.7. The limit operator $\mathcal{L}$ need not satisfy (A_μ), we can recover any operator of the form $\mathcal{L}[\psi] = \mathrm{tr}[\sigma\sigma^T D^2\psi] + \mathcal{L}^\mu[\psi]$; the general form of the generator of a *symmetric Lévy process* [7]. See sections 4.2.2 and 4.5.2 for more details and examples. An extension of this result will be discussed in Section 4.2.3 below.

The stability result will be used along with approximation and compactness arguments to obtain the following existence result and a priori estimates.

Theorem 4.2.8 (Existence and uniqueness). *Assume* (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *Then there exists a unique distributional solution* u *of* (4.1.1) *and* (4.1.2) *satisfying*

$$u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Theorem 4.2.9 (A priori estimates). *Assume* (A_φ), (A_μ), $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *Let* $u, \hat{u}$ *be the distributional solutions of* (4.1.1) *with initial data* $u_0, \hat{u}_0$ *in the sense of Definition 4.2.2 (b) , respectively. Then*

(a) (L^1-contraction) $\int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ \,\mathrm{d}x$, $t \in [0,T]$*;*

(b) (Comparison principle) *If* $u_0 \leq \hat{u}_0$ *a.e. in* $\mathbb{R}^N$, *then* $u \leq \hat{u}$ *a.e. in* Q_T*;*

(c) (L^1-bound) $\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)}$, $t \in [0,T]$*;*

(d) (L^∞-bound) $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}$, $t \in [0,T]$*;*

(e) (Time regularity) *For every* $t, s \in [0,T]$ *and compact set* $K \subset \mathbb{R}^N$,

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1(K)} \leq \lambda_{u_0}\left(|t-s|^{\frac{1}{3}}\right) + C_{K,\varphi,u_0,\mu}\left(|t-s|^{\frac{1}{3}} + |t-s|\right)$$

where $\lambda_{u_0} = \max_{|\sigma|\leq\delta} \|u_0 - u_0(\cdot+\sigma)\|_{L^1(\mathbb{R}^N)}$, $|K|$ *is the Lebesgue measure of* K, *and*

$$C_{K,\varphi,u_0,\mu} = |K| \sup\left\{|\varphi(r)| : |r| \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\right\} \int_{|z|>0} \min\{|z|^2, 1\} \,\mathrm{d}\mu(z).$$

(f) (Mass conservation) *If, in addition, there exists* $L, \delta > 0$ *such that* $|\varphi(r)| \leq L|r|$ *for* $|r| \leq \delta$, *then*

$$\int_{\mathbb{R}^N} u(x,t) \,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x) \,\mathrm{d}x, \quad t \in [0,T].$$

These results are proved in Section 4.4.

Remark 4.2.10. The condition $|\varphi(r)| \leq L|r|$ in Theorem 4.2.9 (f) is sharp in the following sense: If $\varphi(r) = r^m$ for any $m < 1$, then there is $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$ such that positive solutions u of (4.1.1) and (4.1.2) has extinction in finite time and hence $\int u \neq \int u_0$. Simply take $N \in \mathbb{N}$ and $s \in (0,2)$ such that $m \leq \frac{(N-s)^+}{N}$; see [38] for the details.

We now present several applications of the previous results.

4.2.1 Application 1: Compactness, local limits, continuous dependence

We start by a compactness and convergence result for very general approximations of (4.1.1) and (4.1.2).

Theorem 4.2.11 (Compactness and convergence)**.** *Assume $\mathcal{L} : C_c^\infty(Q_T) \to L^1(Q_T)$, μ_n satisfies (A_μ), φ_n and φ satisfy (A_φ), and $u_{0,n} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for every $n \in \mathbb{N}$. Then if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of distributional solutions of (4.2.5) with initial data $\{u_{0,n}\}_{n\in\mathbb{N}}$ in the sense of Definition 4.2.2 (b), and*

(i) $\sup_n \int_{|z|>0} \min\{|z|^2, 1\} \,\mathrm{d}\mu_n(z) < \infty$*;*

(ii) $\sup_n \|u_{0,n}\|_{L^\infty(\mathbb{R}^N)} < \infty$*;*

(iii) $\mathcal{L}^{\mu_n}[\psi] \to \mathcal{L}[\psi]$ *in* $L^1(\mathbb{R}^N)$ *for all* $\psi \in C_c^\infty(\mathbb{R}^N)$*;*

(iv) $\varphi_n \to \varphi$ *locally uniformly;*

(v) $u_{0,n} \to u_0$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$*.*

Then

(a) there exist a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ and a $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ such that

$$u_{n_j} \to u \qquad \text{in} \qquad C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad j \to \infty;$$

(b) the limit u from part (a) is a distributional solution of (4.2.6) and (4.1.2).

The proof can be found in Section 4.5.1. Using this result, we study the case $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, $s \in (0,2)$. As expected, we find that solutions of the fractional equation (4.1.4) converge as $s \to 2^-$ to the solution of the local equation (4.1.5). Then we obtain a new result about continuous dependence on (m, s) for the porous medium equation of [38], that is, equation (4.1.6).

Corollary 4.2.12. *Assume (A_φ) and $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.*

(a) The distributional solution u_s of (4.1.4) and (4.1.2), converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ as $s \to 2^-$ to a function u, and u is a distributional solution of (4.1.5) and (4.1.2).

(b) Let u_n and $\bar{u}$ be distributional solutions of (4.1.6) *and* (4.1.2) *with* $(m,s) = (m_n, s_n)$ *and* $(m,s) = (\bar{m}, \bar{s})$ *respectively. If*

$$(0,\infty) \times (0,2) \ni (m_n, s_n) \quad \longrightarrow \quad (\bar{m}, \bar{s}) \in (0,\infty) \times (0,2],$$

then $u_n \to \bar{u}$ *in* $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$.

The proof of this result can also be found in section 4.5.1.

Remark 4.2.13. When $u_0 \in L^1(\mathbb{R}^N)$, the authors of [38] show continuous dependence in $C([0,T]; L^1(\mathbb{R}^N))$ for (4.1.6) and (4.1.2) for $(m,s) \in \left(\frac{(N-s)^+}{N}, \infty\right) \times (0,2]$. When $m \leq \frac{(N-s)^+}{N}$, we are in the fast diffusion range, and as far as we know, Corollary 4.2.12 (b) provides the first continuous dependence result for this case.

4.2.2 Application 2: Numerical approximation, convergence, existence

Surprisingly, our class of operators $\mathcal{L}^\mu$ is so wide that it contains a lot of its own numerical discretizations! It even contains common discretizations of local operators as well. We illustrate this by giving one such discretization, a basic and very natural one, and then analyzing the resulting semidiscrete numerical method for (4.1.1), or rather (4.2.7). We prove that it satisfies many properties including convergence, and conclude a second and more general existence result. Consider

$$\partial_t u - (L^\sigma + \mathcal{L}^\mu)[\varphi(u)] = 0 \qquad \text{in} \qquad Q_T, \tag{4.2.7}$$

where $\mathcal{L}^\mu$ is defined as before and L^σ is a possibly degenerate local operator

$$L^\sigma[\psi](x) := \operatorname{tr}\big[\sigma\sigma^T D^2\psi(x)\big]$$

where $\sigma = (\sigma_1, ..., \sigma_P) \in \mathbb{R}^{N\times P}$, $P \in \mathbb{N}$, and $\sigma_i \in \mathbb{R}^N$. Note that $L^\sigma + \mathcal{L}^\mu$ is the generator of a *symmetric* Lévy process, and conversely, any *symmetric* Lévy processes has a generator like $L^\sigma + \mathcal{L}^\mu$ (cf. [7]). Moreover, equation (4.1.1) and (4.1.5) are special cases of (4.2.7) since σ and μ may be degenerate or even zero.

For any $h > 0$, we approximate (4.2.7) in the following way,

$$\partial_t u_h - \left(L^\sigma_h + \mathcal{L}^\mu_h\right)[\varphi(u_h)] = 0 \qquad \text{in} \qquad Q_T. \tag{4.2.8}$$

where

$$L_h^\sigma[\psi](x) := \sum_{i=1}^{P} \frac{\psi(x+\sigma_i h)+\psi(x-\sigma_i h)-2\psi(x)}{h^2}, \tag{4.2.9}$$

$$\mathcal{L}_h^\mu[\psi](x) := \sum_{\alpha\neq 0} \left(\psi(x+z_\alpha)-\psi(x)\right)\mu\left(z_\alpha+R_h\right), \tag{4.2.10}$$

and $z_\alpha = h\alpha$, $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}^N$, $R_h = \frac{h}{2}[-1,1)^N$. This is a finite difference approximation of L^σ and quadrature approximation of $\mathcal{L}^\mu$.

Remark 4.2.14. (a) When $\sigma = e_i$, a standard basis vector of $\mathbb{R}^N$, then $L^{e_i} = \frac{\partial_i^2}{\partial x_i^2}$ and $L_h^{e_i}\psi(x) = \frac{\psi(x+he_i)-2\psi(x)+\psi(x-he_i)}{h^2}$; a classical finite difference approximation.

(b) Both L_h^σ and $\mathcal{L}_h^\mu$ are in form (4.1.3) and satisfy (A_μ); cf. Lemma 4.5.2 and 4.5.3.

(c) $L^\sigma\psi(x) = \sum_{i=1}^P \sigma_i^T D^2\psi(x)\sigma_i = \sum_{i=1}^P (\sigma_i^T D)^2\psi(x) \approx L_h^\sigma\psi(x)$.

(d) $\mathcal{L}^\mu[\psi](x) = \sum_{\alpha\in\mathbb{Z}^N}\int_{z_\alpha+R_h}\psi(x+z)-\psi(x)\,\mathrm{d}\mu(z) \approx \mathcal{L}_h^\mu[\psi](x)$.

(e) To avoid $\mu(R_h)$ which may be infinite, we do not sum over $\alpha = 0$ in $\mathcal{L}_h^\mu$.

We now show that the scheme has many good properties, including convergence.

Proposition 4.2.15 (Properties of approximation)**.** *Assume* (A_φ), (A_μ), $\sigma \in \mathbb{R}^{N\times P}$, $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, *and* $h > 0$.

(a) (Existence and uniqueness) *There exists a unique distributional solution* $u_h \in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *of* (4.2.8) *and* (4.1.2).

(b) (L^p-stable) $\|u_h(\cdot,t)\|_{L^p(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{p-1}{p}}\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}, \quad p\in[1,\infty],\ t\in[0,T]$.

(c) (L^1-consistent) *For all* $\psi \in C_c^\infty(\mathbb{R}^N)$

$$\|\left(L_h^\sigma+\mathcal{L}_h^\mu\right)[\psi]-\left(L^\sigma+\mathcal{L}^\mu\right)[\psi]\|_{L^1(\mathbb{R}^N)} \to 0 \qquad as \qquad h\to 0^+.$$

(d) (Monotone) *If* $u_0 \leq \hat{u}_0$ *a.e. in* $\mathbb{R}^N$, *then* $u_h \leq \hat{u}_h$ *a.e. in* Q_T.

(e) (Conservative) *If in addition, there exists* $\delta, L > 0$ *such that* $|\varphi(r)| \leq L|r|$ *for* $|r|\leq\delta$, *then for all* $t\in[0,T]$

$$\int_{\mathbb{R}^N} u_h(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x.$$

Proposition 4.2.16 (Compactness of approximation)**.** *Assume* (A_φ), (A_μ), $\sigma\in\mathbb{R}^{N\times P}$, $u_0 \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, *and* $h>0$. *Then there is subsequence of distributional solutions*

u_h of (4.2.8) and (4.1.2) that converges in $C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ as $h\to 0^+$ to some function u. Moreover, $u\in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ and u is a distributional solution of (4.2.7) and (4.1.2).

Note that Theorem 4.2.16 also provide a new existence result:

Corollary 4.2.17 (Existence for (4.2.7))**.** *Under the assumptions of Proposition 4.2.16, there exists a distributional solution $u\in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ of (4.2.7) and (4.1.2).*

In many cases we can combine the compactness result with uniqueness results for the limit equations, and hence obtain convergence for the approximation.

Theorem 4.2.18 (Convergence of approximation)**.** *Under the assumptions of Proposition 4.2.16, and if in addition either $\sigma\equiv 0$ or $\mu\equiv 0$ and $\sigma=I$ (the identity matrix), then the distributional solutions u_h of (4.2.8) and (4.1.2) converges in $C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ as $h\to 0^+$ to the unique distributional solution $u\in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ of (4.2.7) and (4.1.2).*

The proofs will be given in Section 4.5.2.

Remark 4.2.19. (a) Our approximation is well-defined and converge for *any* problem of the type (4.2.7), including strongly degenerate Stefan problems and fast diffusion equations. The scheme and convergence result thus cover cases that have not been considered before in the literature. For nonlocal problems of this type, there are very few results, and only for locally Lipschitz φ [33, 40, 41].

(b) To obtain a fully discrete numerical method, it remains to (i) restrict the method to some spacial grid and (ii) discretize also in time. Time discretization is easier and leads to a problem that no longer has the form (4.1.1); we will discuss it in a future work. Restriction to a spacial grid can always be done after a change of coordinate system; see Section 4.2.3 below.

(c) The existence result is a result where existence for problems involving nonlocal operators $\mathcal{L}^\mu$ are exported to problems involving the "closure" of this class of operators—namely, operators of the form $L^\sigma+\mathcal{L}^\mu$. The proof is completely different from proofs based on nonlinear semigroup theory; see e.g. Chp. 10 in [77], and [38].

4.2.3 Remarks and extensions

Alternative definition of distributional solutions

(1) A more compact form that we will use in the proofs is the following:

Lemma 4.2.20. *Assume* (A_φ), (A_{u_0}), (A_μ) *and* $u \in L^\infty(Q_T)$. *Then* u *is a distributional solution of* (4.1.1) *and* (4.1.2) *if and only if*

$$\int_0^T \int_{\mathbb{R}^N} u(x,t)\partial_t\psi(x,t) + \varphi(u(x,t))\mathcal{L}^\mu[\psi(\cdot,t)](x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x = 0$$

for all $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T))$.

The easy and standard proof is omitted.

About the initial conditions

(2) The solutions provided by Theorem 4.2.8 belong to $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and hence satisfy the initial condition in the strong L^1_{loc}-sense: For all compact $K \subset \mathbb{R}^N$,

$$\int_K |u(x,t) - u_0(x)|\,\mathrm{d}x \to 0 \qquad \text{as} \qquad t \to 0.$$

(3) If the initial conditions are satisfied in the strong L^1_{loc}-sense, then they are of course also satisfied in the distributional sense of Definition 4.2.2.

Extensions of the uniqueness result Corollary 4.2.4

(4) With the same proof, we also get uniqueness for the initial value problem for the inhomogenenous equation

$$\partial_t u + \mathcal{L}^\mu[\varphi(u)] = g(x,t).$$

(5) A close inspection of the proof reveals that we can replace continuity of φ in (A_φ) by continuity at zero, Borel measurability, and $\varphi(u) \in L^\infty(Q_T)$ (cf. [19]).

Extensions of the stability result Theorem 4.2.6

(6) When φ_n is independent of n, we only need weak convergence of $\mathcal{L}^{\mu_n}$ in (i):

$$\mathcal{L}^{\mu_n}[\psi] \to \mathcal{L}[\psi] \quad \textit{weakly} \text{ in } L^1(\mathbb{R}^N) \text{ for all } \psi \in C_\mathrm{c}^\infty(Q_T).$$

Moreover, by considering subsequences we can replace (iii) by $u_n \to u$ in $L^1_{\mathrm{loc}}(Q_T)$. These observations follow by slight changes in the proof of Theorem 4.2.6 in Section 4.4.

(7) A general condition for L^1-weak convergence of $\mathcal{L}^{\mu_n}$ [28]: There exist $\sigma \in \mathbb{R}^{N\times P}$ and a nonnegative Radon measure μ such that for all $A \in \mathbb{R}^{N\times N}$

(i) $\sup_n \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu_n(z) < \infty$;

(ii) $\int_{|z|\leq 1} zAz^T\,\mathrm{d}\mu_n(z) \to \mathrm{tr}\left(\sigma\sigma^T A\right) + \int_{|z|\leq 1} zAz^T\,\mathrm{d}\mu(z)$;

(iii) $\int_{|z|>1} \mathrm{d}\mu_n(z) \to \int_{|z|>1} \mathrm{d}\mu(z)$.

Here $\mathcal{L} = \mathrm{tr}[\sigma\sigma^T D^2] + \mathcal{L}^\mu$; see [28] for a general discussion and more examples.

Defining the scheme (4.2.8) on a grid

(8) By a coordinate transformation $x = Ay$, $L^\sigma + \mathcal{L}^\mu$ can be transformed into

$$L^{I_0} + \mathcal{L}^{\tilde{\mu}} \qquad \text{where} \qquad I_0 := \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathbb{R}^{N\times N},$$

I is an identity matrix, and $\mathrm{d}\tilde{\mu}(z) = \mathrm{d}\mu(A^{-1}z)$ satisfies (A_μ). Up to permutations of the components of y, $A = QJ$ where $Q \in \mathbb{R}^{N\times N}$ is orthonormal, $Q\sigma\sigma^T Q^T = \mathrm{diag}(\lambda_i)$ for $\lambda_i \geq 0$, and $J = \mathrm{diag}(\sqrt{c_i})$ where $c_i = 1$ if $\lambda_i = 0$ and $c_i = \frac{1}{\lambda_i}$ if $\lambda_i > 0$ for $i = 1, \dots, N$.

(9) For the new operator $L^{I_0}+\mathcal{L}^{\tilde{\mu}}$, our approximations produce an operator $L_h^{I_0}+\mathcal{L}_h^{\tilde{\mu}}$ that can be restricted to the (y-)grid $\mathcal{G}_h := h\mathbb{Z}^N$ ($h > 0$), that is $L_h^{I_0} + \mathcal{L}_h^{\tilde{\mu}} : \mathbb{R}^{\mathcal{G}_h} \to \mathbb{R}^{\mathcal{G}_h}$ is well-defined.

4.3 The proof of uniqueness

4.3.1 Preliminary results

A crucial part in the proof is played by the following linear elliptic equation

$$\varepsilon v_\varepsilon(x) - \mathcal{L}^\mu[v_\varepsilon](x) = g(x) \quad \text{in} \quad \mathbb{R}^N, \tag{4.3.1}$$

where $\varepsilon > 0$ and $\mathcal{L}^\mu$ defined by (4.1.3). Its solutions will be denoted by

$$B_\varepsilon^\mu[g](x) := v_\varepsilon(x).$$

Formally, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of $\mathcal{L}^\mu$. Note that $\mathcal{L}^\mu$ may be very degenerate and therefore Fourier techniques do not easily apply (cf. Example 4.1 and Remark 4.3.8

(a) below). The main results about equation (4.3.1) are given below, while most of the proofs will be given in Section 4.6. Note that in [19] such results are easy in view of an explicit representation formula for B^μ_ε. Here, on the other hand, they are not easy and we have to work quite a lot to prove these estimates. The method of proof is different, more nonlocal, and requires less of the operator.

Theorem 4.3.1 (Classical and distributional solutions)**.** *Assume* (A_μ) *and* $\varepsilon > 0$.

(a) If $g \in C_\mathrm{b}^\infty(\mathbb{R}^N)$*, then there exists a unique classical solution* $B^\mu_\varepsilon[g] \in C_\mathrm{b}^\infty(\mathbb{R}^N)$ *of* (4.3.1)*. Moreover, for each multiindex* $\alpha \in \mathbb{N}^N$*,*

$$\varepsilon \|D^\alpha B^\mu_\varepsilon[g]\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty}.$$

(b) If $g \in L^1(\mathbb{R}^N)$*, then there exists a unique distributional solution* $B^\mu_\varepsilon[g] \in L^1(\mathbb{R}^N)$ *of* (4.3.1)*. Moreover,*

$$\varepsilon \|B^\mu_\varepsilon[g]\|_{L^1(\mathbb{R}^N)} \leq \|g\|_{L^1(\mathbb{R}^N)}.$$

(c) If $g \in L^\infty(\mathbb{R}^N)$*, then there exists a unique distributional solution* $B^\mu_\varepsilon[g] \in L^\infty(\mathbb{R}^N)$ *of* (4.3.1)*. Moreover,*

$$\varepsilon \|B^\mu_\varepsilon[g]\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}.$$

Remark 4.3.2. If $g \in L^1 \cap L^\infty$, then $\varepsilon \|B^\mu_\varepsilon[g]\|_{L^p} \leq \|g\|_{L^\infty}^{\frac{p-1}{p}} \|g\|_{L^1}^{\frac{1}{p}}$ for any $p \in (1, \infty)$.

When a smooth g depends also on time, then $B^\mu_\varepsilon[g]$ will be smooth in time and space.

Corollary 4.3.3. *Assume* (A_μ)*,* $\varepsilon > 0$*, and* $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T))$*. Then*

(a) $B^\mu_\varepsilon[\gamma] \in C_\mathrm{b}^\infty(\mathbb{R}^N \times [0,T))$*.*

(b) $B^\mu_\varepsilon[\gamma](x,.)$ *is compactly supported in* $[0,T)$*.*

(c) $\partial_t(B^\mu_\varepsilon[\gamma]) = B^\mu_\varepsilon[\partial_t\gamma]$ *and* $B^\mu_\varepsilon[\gamma], B^\mu_\varepsilon[\partial_t\gamma], \mathcal{L}^\mu\left[B^\mu_\varepsilon[\gamma]\right] \in L^1(Q_T)$*.*

Proof. (a) A standard argument using difference quotients, linearity and uniqueness of the problem, the L^∞-bound of Theorem 4.3.1 (a), and induction on n, gives that

$$\partial_t^n D^\alpha B^\mu_\varepsilon[\gamma] = B^\mu_\varepsilon[\partial_t^n D^\alpha \gamma] \qquad \text{in} \qquad Q_T \tag{4.3.2}$$

for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$. This argument is almost exactly the same as the one given in the proof of Proposition 4.6.8 (d) below. Then by Theorem 4.3.1 (a),

$$\varepsilon \|\partial_t^n D^\alpha B^\mu_\varepsilon[\gamma]\|_{L^\infty(Q_T)} \leq \|\partial_t^n D^\alpha \gamma\|_{L^\infty(Q_T)}.$$

(b) Holds since B^μ_ε is an operator in the spatial variable x and $B^\mu_\varepsilon[0] = 0$.

(c) Note that $\partial_t B^\mu_\varepsilon[\gamma] = B^\mu_\varepsilon[\partial_t\gamma]$ by (4.3.2), and by Theorem 4.3.1 (b) and the time continuity of γ and $B^\mu_\varepsilon[\gamma]$,

$$\varepsilon\|B^\mu_\varepsilon[\gamma]\|_{L^1(Q_T)} \leq \|\gamma\|_{L^1(Q_T)},$$

which is finite because $\gamma \in C^\infty_{\mathrm{c}}(Q_T)$. Hence it follows that

$$\varepsilon\|\partial_t(B^\mu_\varepsilon[\gamma])\|_{L^1(Q_T)} = \varepsilon\|B^\mu_\varepsilon[\partial_t\gamma]\|_{L^1(Q_T)} \leq \|\partial_t\gamma\|_{L^1(Q_T)},$$

By equation (4.3.1), $\mathcal{L}^\mu[B^\mu_\varepsilon[\gamma]] = \varepsilon B^\mu_\varepsilon[\gamma] - \gamma$ for all $(x,t) \in Q_T$. Since both $B^\mu_\varepsilon[\gamma]$ and γ are in $L^1(Q_T)$, it follows that also $\mathcal{L}^\mu[B^\mu_\varepsilon[\gamma]] \in L^1(Q_T)$. □

The operator B^μ_ε is self-adjoint in the following sense:

Lemma 4.3.4. *Assume* (A_μ), $g \in L^\infty(\mathbb{R}^N)$, $f \in L^1(\mathbb{R}^N)$, *and* $\varepsilon > 0$. *Then*

$$\int_{\mathbb{R}^N} B^\mu_\varepsilon[g](x) f(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} g(x) B^\mu_\varepsilon[f](x)\,\mathrm{d}x.$$

The proof is given in section 4.6. To prove these and other results in this chapter, we will need some properties of the nonlocal operator $\mathcal{L}^\mu$ that are given below.

Lemma 4.3.5. *Assume* (A_μ).

(a) If $\psi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *then*

$$|\mathcal{L}^\mu[\psi](x)| \leq \frac{1}{2}\max_{|z|\leq 1}|D^2\psi(x+z)|\int_{|z|\leq 1}|z|^2\,\mathrm{d}\mu(z) + 2\|\psi\|_{L^\infty(\mathbb{R}^N)}\int_{|z|>1}\mathrm{d}\mu(z).$$

(b) Let $p \in \{1,\infty\}$ *be fixed. If* $\psi \in W^{2,p}(\mathbb{R}^N)$, *then*

$$\|\mathcal{L}^\mu[\psi]\|_{L^p(\mathbb{R}^N)} \leq \frac{1}{2}\|D^2\psi\|_{L^p(\mathbb{R}^N)}\int_{|z|\leq 1}|z|^2\,\mathrm{d}\mu(z) + 2\|\psi\|_{L^p(\mathbb{R}^N)}\int_{|z|>1}\mathrm{d}\mu(z).$$

(c) If $\psi_1 \in W^{2,1}(\mathbb{R}^N)$ *and* $\psi_2 \in W^{2,\infty}(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N}\psi_1\mathcal{L}^\mu[\psi_2]\,\mathrm{d}x = \int_{\mathbb{R}^N}\mathcal{L}^\mu[\psi_1]\psi_2\,\mathrm{d}x.$$

Remark 4.3.6. (a) If $\psi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $\mathcal{L}^\mu[\psi](x)$ is well-defined by (a).

(b) If $\mu(\mathbb{R}^N) < \infty$, a density argument and symmetry of μ reveals that

$$\mathcal{L}^\mu[\phi](x) = \int_{|z|>0}\phi(x+z) - \phi(x)\ \mathrm{d}\mu(z),$$

and the assumptions of Lemma 4.3.4 can be relaxed to $\psi \in L^\infty(\mathbb{R}^N)$, $\psi \in L^p(\mathbb{R}^N)$ for $p \in \{1, \infty\}$, and $\psi_1 \in L^1(\mathbb{R}^N)$ and $\psi_2 \in L^\infty(\mathbb{R}^N)$ respectively in (a), (b), and (c). The second derivative part of the estimates in (a) and (b) then have to be dropped and the remaining term modified accordingly.

A proof of Lemma 4.3.5 can be found e.g. in Sections 1 and 4 in [3].

Lemma 4.3.7. *Assume* (A_μ) *and* $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. *Then*

$$\mathcal{F}(\mathcal{L}^\mu[\psi])(\xi) = -\sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(\psi)(\xi),$$

where

$$\sigma_{\mathcal{L}^\mu}(\xi) := \int_{|z|>0} 1 - \cos(z \cdot \xi) \,\mathrm{d}\mu(z).$$

Moreover, $\sigma_{\mathcal{L}^\mu}(\xi) \geq 0$ *and*

$$\Big(\psi, \mathcal{L}^\mu[\psi]\Big)_{L^2(\mathbb{R}^N)} = -\Big\|(\mathcal{L}^\mu)^{\frac{1}{2}}[\psi]\Big\|^2_{L^2(\mathbb{R}^N)}.$$

Remark 4.3.8. (a) $\sigma_{\mathcal{L}^\mu}$ is the Fourier symbol of $\mathcal{L}^\mu$. In our generality it may not be invertible or have any smoothing properties. An extreme example is $\mu = \delta_{z_0}$ for $z_0 \neq 0$, where $\sigma_{\mathcal{L}^\mu}(\xi) = 1 - \cos z_0 \cdot \xi$; a bounded function with infinitly many zeros.

(b) If $\psi, \mathcal{L}^\mu[\psi] \in L^2(\mathbb{R}^N)$, then a density argument shows that the Fourier symbol exists and the conclusions of Lemma 4.3.7 still hold.

(c) The notation $(\mathcal{L}^\mu)^{\frac{1}{2}}$ is used to denote the square root of the operator $\mathcal{L}^\mu$ in the Fourier transform sense.

Proof. By the definition of $\mathcal{L}^\mu$, Fubini's theorem, and the symmetry of μ,

$$\begin{aligned}
\mathcal{F}(\mathcal{L}^\mu[\psi])(\xi) &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}x\cdot\xi} \int_{|z|>0} \psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1} \,\mathrm{d}\mu(z) \,\mathrm{d}x \\
&= \int_{|z|>0} \mathrm{e}^{\mathrm{i}z\cdot\xi}\mathcal{F}(\psi)(\xi) - \mathcal{F}(\psi)(\xi) - \mathrm{i}z \cdot \xi \mathbf{1}_{|z|\leq 1}\mathcal{F}(\psi)(\xi) \,\mathrm{d}\mu(z) \\
&= \mathcal{F}(\psi)(\xi) \int_{|z|>0} \cos(z \cdot \xi) - 1 \,\mathrm{d}\mu(z).
\end{aligned}$$

To show the second part of the lemma, note that $\sigma_{\mathcal{L}^\mu} \geq 0$ and $\psi, \mathcal{L}^\mu[\psi] \in L^2(\mathbb{R}^N)$ (cf. Lemma 4.3.5 (b)). It follows that $\mathcal{F}(\psi), \sigma_{\mathcal{L}^\mu}\mathcal{F}(\psi) \in L^2(\mathbb{R}^N)$, and then by the inequality $2ab \leq a^2 + b^2$, $(\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}}\mathcal{F}(\psi) \in L^2(\mathbb{R}^N)$. By Plancherel's theorem,

$$\begin{aligned}
\Big(\psi, \mathcal{L}^\mu[\psi]\Big)_{L^2(\mathbb{R}^N)} &= \Big(\mathcal{F}(\psi), \mathcal{F}(\mathcal{L}^\mu[\psi])\Big)_{L^2(\mathbb{R}^N)} = \Big(\mathcal{F}(\psi), -\sigma_{\mathcal{L}^\mu}\mathcal{F}(\psi)\Big)_{L^2(\mathbb{R}^N)} \\
&= -\Big((\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}}\mathcal{F}(\psi), (\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}}\mathcal{F}(\psi)\Big)_{L^2(\mathbb{R}^N)} = -\Big\|(\mathcal{L}^\mu)^{\frac{1}{2}}[\psi]\Big\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}$$

which completes the proof. □

The following theorem is a key technical tool in our uniqueness argument.

Theorem 4.3.9. *Assume* (A_μ) *and* supp$\mu \neq \emptyset$*. If* $v \in C_0(\mathbb{R}^N)$ *solves*

$$\mathcal{L}^\mu[v] = 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N),$$

then $v \equiv 0$ *for all* $x \in \mathbb{R}^N$.

We give the proof of Theorem 4.3.9 in Appendix 4.7. In the local case [19] such a result follows for example from the Liouville theorem for the Laplacian. Our result is much weaker since we need to ask for some kind of decay at infinity. But on the other hand, Theorem 4.3.9 covers very degenerate operators $\mathcal{L}^\mu$ which do not satisfy any sort of Liouville theorem.

Example 4.1. *Let* $\mu = \delta_{2\pi} + \delta_{-2\pi}$*. Note that* ($\mathrm{A}_\mu$) *holds and that for smooth functions* v*,*

$$\mathcal{L}^\mu[v](x) = v(x+2\pi) - 2v(x) + v(x-2\pi).$$

The function $v = \cos \in C_\mathrm{b}^\infty(\mathbb{R})$ *is an example of a nonconstant function that satisfies* $\mathcal{L}^\mu[v](x) = 0$ *in* $\mathbb{R}$*, and hence the Liouville theorem does not hold for* $\mathcal{L}^\mu$*.*

4.3.2 The proof of Theorem 4.2.3

We define

$$U(x,t) := u(x,t) - \hat{u}(x,t) \qquad \text{and} \qquad \Phi(x,t) := \varphi(u(x,t)) - \varphi(\hat{u}(x,t)).$$

By the assumptions (4.2.1), (4.2.2), and (A_φ),

$$U \in L^1(Q_T) \cap L^\infty(Q_T), \quad \Phi \in L^\infty(Q_T),$$

and by (4.2.3), (4.2.4), and Lemma 4.2.20

$$\int_0^T \int_{\mathbb{R}^N} U\partial_t\psi + \Phi\mathcal{L}^\mu[\psi]\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \text{for all} \qquad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T)). \tag{4.3.3}$$

We emphasize that this equation also incorporates a zero intitial condition for U.

We now define the function $h_\varepsilon(t)$ which will play the main role in the proof:

$$h_\varepsilon(t) := (B_\varepsilon^\mu[U](\cdot,t), U(\cdot,t)) = \int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot,t)](x)U(x,t)\,\mathrm{d}x. \tag{4.3.4}$$

Note that $h_\varepsilon \in L^1((0,T))$ since $\|h_\varepsilon\|_{L^1((0,T))} \leq \frac{1}{\varepsilon}\|U\|_{L^\infty(Q_T)}\|U\|_{L^1(Q_T)}$ by Theorem 4.3.1 (b). For the proof of Theorem 4.2.3, we will now show that there is a sequence $\varepsilon_n \to 0^+$ such that $\lim_{n\to\infty} h_{\varepsilon_n}(t) = 0$. To do that we start by the following lemma:

Lemma 4.3.10. *Assume* (A_μ), $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$, *and* (4.3.3) *holds. Then*

(a) $\displaystyle\iint_{Q_T} \Big(B^\mu_\varepsilon[U]\partial_t\psi + (\varepsilon B^\mu_\varepsilon[\Phi] - \Phi)\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0$ *for all* $\psi \in C^\infty_\mathrm{c}(\mathbb{R}^N \times [0,T))$.

(b) $\displaystyle B^\mu_\varepsilon[U(\cdot,t)](x) = \int_0^t \Big(\varepsilon B^\mu_\varepsilon[\Phi(\cdot,s)](x) - \Phi(x,s)\Big)\,\mathrm{d}s$ *a.e.* $(x,t) \in \mathbb{R}^N \times (0,T)$.

(c) For a.e. $t \in (0,T)$, $\|B^\mu_\varepsilon[U](\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq 2t\|\Phi\|_{L^\infty(Q_T)}$.

Proof. (a) We fix $\gamma \in C^\infty_\mathrm{c}(\mathbb{R}^N \times [0,T))$ and take $\psi = B^\mu_\varepsilon[\gamma]$ as a test function in (4.3.3). Note that ψ is an admissible test function by a density argument using Corollary 4.3.3 (a)–(c) and $U, \Phi \in L^\infty(Q_T)$. Then by (4.3.1) and Corollary 4.3.3 (c),

$$0 = \iint_{Q_T} U\partial_t(B^\mu_\varepsilon[\gamma]) + \Phi\mathcal{L}^\mu\left[B^\mu_\varepsilon[\gamma]\right]\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} UB^\mu_\varepsilon[\partial_t\gamma] + \Phi\Big(\varepsilon B^\mu_\varepsilon[\gamma] - \gamma\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, the self-adjointness of B^μ_ε (cf. Lemma 4.3.4) yields

$$\int_0^T\int_{\mathbb{R}^N} B^\mu_\varepsilon[U]\partial_t\gamma + \Big(\varepsilon B^\mu_\varepsilon[\Phi] - \Phi\Big)\gamma\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which completes the proof.

(b) This result follows from (a) and a special choice of test function. For $0 < s < T$, $a > 0$, and $0 < \delta < T - a$, we define

$$\theta_a(t) = \begin{cases} 1 & t \leq s-a \\ 1 - \frac{1}{a}(t-s+a) & s-a < t < s \\ 0 & t \geq s \end{cases} \qquad \text{and} \qquad \theta_{a,\delta}(t) = \theta_a * \rho_\delta(t),$$

where the mollifier ρ_δ is defined in (4.1.8). Then $\theta_{a,\delta} \in C^\infty_\mathrm{b}((0,T)) \cap L^1((0,T))$ and $\mathrm{supp}\{\theta_{a,\delta}\} \subset [-\infty, T)$. Let $\gamma \in C^\infty_\mathrm{c}(\mathbb{R}^N)$ and take $\psi(x,t) = \theta_{a,\delta}(t)\gamma(x) \in C^\infty_\mathrm{c}(\mathbb{R}^N \times [0,T))$ as a test function in part (a). Then we use properties of mollifiers and Lebesgue's dominated convergence theorem to send $\delta \to 0^+$ and get

$$\iint_{Q_T} \Big(B^\mu_\varepsilon[U]\theta'_a + (\varepsilon B^\mu_\varepsilon[\Phi] - \Phi)\theta_a\Big)\gamma\,\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By Fubini's theorem and since $\theta_a'(t) = -\frac{1}{a}\mathbf{1}_{s-a<t<s}$ and $\operatorname{supp}\{\theta_a\} = [0, s]$, we find that

$$\int_{\mathbb{R}^N} \left(\frac{1}{a}\int_{s-a}^{s} B_\varepsilon^\mu[U]\,\mathrm{d}t + \int_0^s (\varepsilon B_\varepsilon^\mu[\Phi] - \Phi)\theta_a\,\mathrm{d}t\right)\gamma\ \mathrm{d}x = 0.$$

We now send $a \to 0^+$. Since $\int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot,t)](x)\gamma(x)\,\mathrm{d}x \in L^1(0,T)$ by Fubini's theorem,

$$\frac{1}{a}\int_{s-a}^{s}\int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot,t)](x)\gamma(x)\,\mathrm{d}x\,\mathrm{d}t \to \int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot,s)](x)\gamma(x)\,\mathrm{d}x \quad \text{as} \quad a \to 0^+$$

for a.e. s by Lebesgue's differentiation theorem. For the other term, we may use Lebesgue's dominated convergence theorem to pass to the limit. Since $\theta_a \to \mathbf{1}_{[0,s)}$ pointwise, we find that for a.e. $s \in [0,T]$,

$$\int_{\mathbb{R}^N} \left(B_\varepsilon^\mu[U(\cdot,s)](x) + \int_0^s (\varepsilon B_\varepsilon^\mu[\Phi(\cdot,t)](x) - \Phi(x,t))\,\mathrm{d}t\right)\gamma(x)\ \mathrm{d}x = 0.$$

Since $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$ is arbitrary, part (b) follows.

(c) By part (b) and Theorem 4.3.1 (c), $\|B_\varepsilon^\mu[U](\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq 2t\|\Phi\|_{L^\infty(Q_T)}$ a.e. □

Proposition 4.3.11. *Assume* (A_μ), $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$, *and* (4.3.3) *holds. Then* $h_\varepsilon(t)$ *defined by* (4.3.4) *is absolutely continuous and*

$$h_\varepsilon'(t) = 2\big(\varepsilon B_\varepsilon^\mu[\Phi](\cdot,t) - \Phi(\cdot,t), U(\cdot,t)\big) \qquad \textit{in} \qquad \mathcal{D}'\left((0,T)\right).$$

The proof below is an adaptation of the proof in [19, pp. 157–158].

Proof. Let the mollifier $\rho_\delta = \rho_\delta(t)$ be defined in (4.1.8), the extension $\bar{U}$ be U on Q_T and zero outside Q_T, and

$$\bar{U}_\delta(x,t) := \bar{U}(x,\cdot) * \rho_\delta(t) = \int_{\mathbb{R}} \bar{U}(x,s)\rho_\delta(t-s)\,\mathrm{d}s.$$

By Young's inequality, $\|\bar{U}_\delta\|_{L^\infty(Q_T)} \leq \|U\|_{L^\infty(Q_T)}$ and $\|\bar{U}_\delta\|_{L^1(Q_T)} \leq \|U\|_{L^1(Q_T)}$. Moreover, the time continuity of $\bar{U}_\delta$, Corollary 4.3.3 (c), and Lemma 4.3.4 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} B_\varepsilon^\mu[\bar{U}_\delta]\bar{U}_\delta\,\mathrm{d}x = 2\int_{\mathbb{R}^N} \partial_t\left(B_\varepsilon^\mu\left[\bar{U}_\delta\right]\right)\bar{U}_\delta\,\mathrm{d}x = 2\int_{\mathbb{R}^N} \partial_t(\bar{U}_\delta)B_\varepsilon^\mu[\bar{U}_\delta]\,\mathrm{d}x \tag{4.3.5}$$

for $t \in \mathbb{R}$.

Let us show that

$$B_\varepsilon^\mu[\bar{U}_\delta(\cdot,t)](x) = \int_{\mathbb{R}} B_\varepsilon^\mu[\bar{U}(\cdot,s)](x)\rho_\delta(t-s)\,\mathrm{d}s \qquad \text{in} \qquad Q_T. \tag{4.3.6}$$

First assume that $\bar{U} \in C_\mathrm{b}^\infty(Q_T) \cap L^1(Q_T)$. Then $B_\varepsilon^\mu[\bar{U}(\cdot,t)] \in C_\mathrm{b}^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for $t \in [0,T]$, and thus, it solves (4.3.1) pointwise in $\mathbb{R}^N$. Multiply this equation by $\rho_\delta(s-t)$, integrate over $\mathbb{R}$, and use Fubini's theorem and the uniqueness in Theorem 4.3.1 (b) and (c) to find that (4.3.6) holds. A density/mollification argument using uniqueness and $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ estimates from Theorem 4.3.1 then shows that (4.3.6) also holds (a.e.!) for $\bar{U} \in L^1(Q_T) \cap L^\infty(Q_T)$.

Let the extension $\bar{\Phi}$ be Φ on Q_T and zero outside Q_T. Using Lemma 4.3.10 (a) with test functions $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times (\delta, T-\delta))$ we get that

$$\partial_t B_\varepsilon^\mu[\bar{U}_\delta(\cdot,t)](x) = \Big(\big(\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}\big)(x,\cdot) * \rho_\delta\Big)(t) \quad \text{a.e. in} \quad \mathbb{R}^N \times (\delta, T-\delta).$$

For any $\Theta \in C_\mathrm{c}^\infty((0,T))$ and sufficiently small δ, we then conclude from (4.3.5) that

$$-\int_0^T \big(B_\varepsilon^\mu[\bar{U}_\delta](\cdot,t), \bar{U}_\delta(\cdot,t)\big)\Theta'(t)\,\mathrm{d}t = 2\int_0^T \big((\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta(t), \bar{U}_\delta(\cdot,t)\big)\Theta(s)\,\mathrm{d}t.$$

By properties of mollifiers and Theorem 4.3.1 (b) and (c),

$$\begin{aligned}
&\bar{U}_\delta \to U \qquad \text{in} \qquad L^1(Q_T),\\
&(\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta \to \varepsilon B_\varepsilon^\mu[\Phi] - \Phi \qquad \text{a.e. in} \qquad Q_T,\\
&\varepsilon\|B_\varepsilon^\mu[\bar{U}_\delta]\|_{L^\infty(Q_T)} \leq \|U\|_{L^\infty(Q_T)},\\
&|(\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta| \leq 2\|\Phi\|_{L^\infty(Q_T)}.
\end{aligned}$$

Now we send $\delta \to 0^+$ using Lebesgue's dominated convergence theorem, and then by the definition of h_ε, we find that

$$-\int_0^T h_\varepsilon(t)\Theta'(t)\,\mathrm{d}t = 2\int_0^T \big(\varepsilon B_\varepsilon^\mu[\Phi](\cdot,t) - \Phi(\cdot,t), U(\cdot,t)\big)\Theta(t)\,\mathrm{d}t.$$

That is, h_ε is weakly differentiable and the weak derivative is

$$h_\varepsilon'(t) = 2\big(\varepsilon B_\varepsilon^\mu[\Phi](\cdot,t) - \Phi(\cdot,t), U(\cdot,t)\big).$$

Moreover, $h_\varepsilon' \in L^1((0,T))$ since by Theorem 4.3.1 (c),

$$\int_0^T |h_\varepsilon'(t)|\,\mathrm{d}t \leq 4\|\Phi\|_{L^\infty(Q_T)}\|U\|_{L^1(Q_T)}.$$

Hence, $h_\varepsilon(t)$ is absolutely continuous, and the proof is complete. □

Proposition 4.3.12. *Assume* (A_φ), (A_μ), $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$ *and* (4.3.3) *holds. Then* *(a) For a.e.* $t \in [0,T]$

$$h_\varepsilon(t) = \varepsilon \|B_\varepsilon^\mu[U](\cdot,t)\|_{L^2}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_\varepsilon^\mu[U]](\cdot,t)\|_{L^2}^2.$$

(b) If a sequence $\varepsilon_n B_{\varepsilon_n}^\mu[U] \to 0$ *a.e. in* Q_T *as* $\varepsilon_n \to 0^+$*, then for a.e.* $t \in [0,T]$*,*

$$\lim_{\varepsilon_n \to 0^+} h_{\varepsilon_n}(t) = 0.$$

We need a technical lemma (cf. [19]).

Lemma 4.3.13. *Assume* (A_φ) *and* (4.2.2)*. Then the Lebesgue measure of the set*

$$S^\xi := \{(x,t) \in Q_T : |\varphi(u(x,t)) - \varphi(\hat{u}(x,t))| > \xi\},$$

is finite for all $\xi > 0$*.*

Proof. Define the set

$$S_u^\delta = \{(x,t) \in Q_T : |u(x,t) - \hat{u}(x,t)| > \delta\}.$$

If $(x,t) \in S^\xi$, then by the continuity of φ there exists a $\delta > 0$ such that $|u(x,t) - \hat{u}(x,t)| > \delta$, that is, $S^\xi \subset S_u^\delta$. By (4.2.2),

$$\delta |S_u^\delta| < \iint_{Q_T} |u(x,t) - \hat{u}(x,t)| \,\mathrm{d}x\,\mathrm{d}t < \infty,$$

and thus, S^ξ also has finite Lebesgue measure. □

Proof of Proposition 4.3.12. (a) By the assumptions, Theorem 4.3.1 (b) and (c), interpolation between $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, and Fubini's theorem, we have for a.e. $t \in [0,T]$ that $U, B_\varepsilon^\mu[U] \in L^2(\mathbb{R}^N)$ and

$$\varepsilon B_\varepsilon^\mu[U] - \mathcal{L}^\mu[B_\varepsilon^\mu[U]] = U \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N). \tag{4.3.7}$$

Hence it follows that $\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]] \in L^2(\mathbb{R}^N)$, where $\overline{\mathcal{L}}^\mu$ is defined through the relation

$$\int_{\mathbb{R}^N} \overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]]\psi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^N} B_\varepsilon^\mu[U]\mathcal{L}^\mu[\psi] \,\mathrm{d}x\,\mathrm{d}t \qquad \text{for all} \qquad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N).$$

Using Plancherel's theorem and Lemma 4.3.7, we then find that for any $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{F}\big(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]]\big)\mathcal{F}(\psi)\,\mathrm{d}\xi &= \int_{\mathbb{R}^N} \mathcal{F}(B_\varepsilon^\mu[U])\mathcal{F}(\mathcal{L}^\mu[\psi])\,\mathrm{d}\xi\\ &= -\int_{\mathbb{R}^N} \mathcal{F}(B_\varepsilon^\mu[U])\sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(\psi)\,\mathrm{d}\xi,\end{aligned}$$

and hence

$$\int_{\mathbb{R}^N} \mathcal{F}(\psi)(\xi)\Big(\mathcal{F}\big(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]]\big)(\xi) + \sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(B_\varepsilon^\mu[U])(\xi)\Big)\,\mathrm{d}\xi = 0.$$

Then by a density argument, we conclude that

$$\mathcal{F}\big(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]]\big)(\xi) = -\sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(B_\varepsilon^\mu[U])(\xi) \qquad \text{in} \qquad L^2(\mathbb{R}^N),$$

and thus, for a.e. $t \in [0,T]$, we have $\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]] = \mathcal{L}^\mu[B_\varepsilon^\mu[U]]$ in $L^2(\mathbb{R}^N)$.

Since $U, B_\varepsilon^\mu[U], \mathcal{L}^\mu[B_\varepsilon^\mu[U]] \in L^2(\mathbb{R}^N)$, equation (4.3.7) holds in $L^2(\mathbb{R}^N)$. By Lemma 4.3.7, Remark 4.3.8 (b), and the definition of h_ε (see (4.3.4)), we have for a.e. $t \in [0,T]$ that

$$\begin{aligned} h_\varepsilon(t) &= \big(B_\varepsilon^\mu[U](\cdot,t), U(\cdot,t)\big)_{L^2(\mathbb{R}^N)}\\ &= \big(B_\varepsilon^\mu[U](\cdot,t), \varepsilon B_\varepsilon^\mu[U](\cdot,t) - \mathcal{L}^\mu[B_\varepsilon^\mu[U]](\cdot,t)\big)_{L^2(\mathbb{R}^N)}\\ &= \varepsilon\|B_\varepsilon^\mu[U](\cdot,t)\|^2_{L^2(\mathbb{R}^N)} - \big(B_\varepsilon^\mu[U](\cdot,t), \mathcal{L}^\mu[B_\varepsilon^\mu[U]](\cdot,t)\big)_{L^2(\mathbb{R}^N)}.\\ &= \varepsilon\|B_\varepsilon^\mu[U](\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_\varepsilon^\mu[U]]\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

(b) By part (a), Proposition 4.3.11, and $U\Phi = (u-\hat{u})(\varphi(u)-\varphi(\hat{u})) \geq 0$,

$$\begin{aligned} 0 \leq h_\varepsilon(t) &= h_\varepsilon(0+) + \int_0^t h_\varepsilon'(s)\,\mathrm{d}s\\ &\leq h_\varepsilon(0+) + 2\int_0^t \big(\varepsilon B_\varepsilon^\mu[\Phi](\cdot,s), U(\cdot,s)\big)\,\mathrm{d}s.\end{aligned} \tag{4.3.8}$$

By the (absolute) continuity of h_ε, Hölder's inequality, Lemma 4.3.10 (c), and Lebesgue's dominated convergence theorem (valid since $U \in L^1(Q_T)$),

$$\begin{aligned} h_\varepsilon(0+) = \lim_{t\to 0^+}\frac{1}{t}\int_0^t h_\varepsilon(s)\,\mathrm{d}s &\leq \lim_{t\to 0^+}\frac{1}{t}\int_0^t \|B_\varepsilon^\mu[U](\cdot,s)\|_{L^\infty(\mathbb{R}^N)}\|U(\cdot,s)\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}s\\ &\leq 2\|\Phi\|_{L^\infty(Q_T)}\lim_{t\to 0^+}\int_0^T \|U(\cdot,s)\|_{L^1(\mathbb{R}^N)}\mathbf{1}_{(0,t)}(s)\,\mathrm{d}s = 0.\end{aligned}$$

Let $\xi > 0$. By the self-adjointness of B_ε^μ (cf. Lemma 4.3.4) and Theorem 4.3.1 (b), we get for a.e. $t \in [0,T]$

$$
\begin{aligned}
\big(\varepsilon B_\varepsilon^\mu[\Phi](\cdot,t), U(\cdot,t)\big) &= \int_{\mathbb{R}^N} \Phi(x,t)\varepsilon B_\varepsilon^\mu[U(\cdot,t)](x)\,\mathrm{d}x\\
&\leq \|\Phi\|_{L^\infty} \int_{\{|\Phi(x,t)|>\xi\}} |\varepsilon B_\varepsilon^\mu[U]|\,\mathrm{d}x + \xi \int_{\{|\Phi(x,t)|\leq\xi\}} |\varepsilon B_\varepsilon^\mu[U]|\,\mathrm{d}x\\
&\leq \|\Phi\|_{L^\infty} \int_{\mathbb{R}^N} |\varepsilon B_\varepsilon^\mu[U(\cdot,t)]|\,\mathbf{1}_{|\Phi(x,t)|>\xi}\,\mathrm{d}x + \xi\|U(\cdot,t)\|_{L^1(\mathbb{R}^N)}.
\end{aligned}
$$

Let t be a point where this inequality holds and $\varepsilon_n B_{\varepsilon_n}^\mu[U(\cdot,t)] \to 0$ a.e. x and $|\varepsilon B_\varepsilon^\mu[U(\cdot,t)](x)| \leq \|U\|_{L^\infty(Q_T)}$ a.e. x (using Theorem 4.3.1 (c)). For any $\eta > 0$, take ξ such that $\xi\|U(\cdot,t)\|_{L^1} < \frac{1}{2}\eta$. Then note that $|\varepsilon B_\varepsilon^\mu[U]|\,\mathbf{1}_{|\Phi(x,t)|>\xi}$ is dominated by $\|U\|_{L^\infty}\mathbf{1}_{|\Phi(x,t)|>\xi}$ which is integrable by Lemma 4.3.13. By Lebesgue's dominated convergence theorem it then follows that $\int_{\mathbb{R}^N} |\varepsilon_n B_{\varepsilon_n}^\mu[U(\cdot,t)]|\mathbf{1}_{|\Phi(x,t)|>\xi}\,\mathrm{d}x < \frac{1}{2}\eta$ when ε_n is small enough. Since this holds for a.e. $t \in [0,T]$, we have proved that

$$
\lim_{\varepsilon_n\to 0^+} \big(\varepsilon_n B_{\varepsilon_n}^\mu[\Phi](\cdot,t), U(\cdot,t)\big) \leq 0 \qquad \text{for a.e.} \qquad t \in [0,T].
$$

We conclude the proof using Lebesgue's dominated convergence theorem to send $\varepsilon_n \to 0^+$ in (4.3.8) (the integrand is dominated by $\|\Phi\|_{L^\infty(Q_T)}\|U(\cdot,t)\|_{L^1(\mathbb{R}^N)} \in L^1((0,T))$ since $U \in L^1(Q_T) \cap L^\infty(Q_T)$). □

Proposition 4.3.14. *Assume* (A_μ)*, supp*$\mu \neq \emptyset$*, and* $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$*. Then there exists a sequence such that* $\varepsilon_n B_{\varepsilon_n}^\mu[g] \to 0$ *a.e. in* $\mathbb{R}^N$ *as* $\varepsilon_n \to 0^+$.

This proposition will be proved later in this section. We are now ready to prove our main result.

Proof of Theorem 4.2.3. In the case that *supp*$\mu = \emptyset$, $\mu \equiv 0$ and $\mathcal{L}^\mu \equiv 0$. Then equation (4.1.1) becomes the ODE $u_t = 0$, and uniqueness follows by standard arguments (e.g. one can easily deduce that $\int_{\mathbb{R}^N} |u(x,t) - \hat{u}(x,t)|\,\mathrm{d}x \leq \int_{\mathbb{R}^N} |u(x,0) - \hat{u}(x,0)|\,\mathrm{d}x$).

Now consider the case *supp*$\mu \neq \emptyset$. By Proposition 4.3.14 and 4.3.12 (a) and (b), there is a sequence such that for a.e. $t \in [0,T]$,

$$
\varepsilon_n\|B_{\varepsilon_n}^\mu[U](\cdot,t)\|_{L^2}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_{\varepsilon_n}^\mu[U]](\cdot,t)\|_{L^2}^2 \to 0 \qquad \text{as} \qquad \varepsilon_n \to 0^+. \tag{4.3.9}
$$

Let $\psi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$. By Plancherel's theorem, Lemma 4.3.7, and Cauchy-Schwarz' inequality, and finally, by (4.3.9), we get for a.e. $t \in [0,T]$ that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} B^{\mu}_{\varepsilon_n}[U]\mathcal{L}^{\mu}[\psi]\,\mathrm{d}x\right| &= \left|-\int_{\mathbb{R}^N} (\mathcal{L}^{\mu})^{\frac{1}{2}}[B^{\mu}_{\varepsilon_n}[U]](\mathcal{L}^{\mu})^{\frac{1}{2}}[\psi]\,\mathrm{d}x\right| \\ &\leq \|(\mathcal{L}^{\mu})^{\frac{1}{2}}[B^{\mu}_{\varepsilon_n}[U]]\|_{L^2(\mathbb{R}^N)}\|(\mathcal{L}^{\mu})^{\frac{1}{2}}[\psi]\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as} \quad \varepsilon_n \to 0^+.\end{aligned}$$

Moreover, by Cauchy-Schwarz' inequality and (4.3.9), we have for a.e. $t \in [0,T]$

$$\left|\int_{\mathbb{R}^N} \varepsilon_n B^{\mu}_{\varepsilon_n}[U]\psi\,\mathrm{d}x\right| \leq \|\varepsilon_n B^{\mu}_{\varepsilon_n}[U]\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as} \quad \varepsilon_n \to 0^+.$$

Hence we conclude that as $\varepsilon_n \to 0^+$,

$$U = \varepsilon_n B^{\mu}_{\varepsilon_n}[U] - \mathcal{L}^{\mu}[B^{\mu}_{\varepsilon_n}[U]] \to 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N),$$

for a.e. $t \in [0,T]$. That is,

$$u - \hat{u} = U \equiv 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N)$$

for a.e. $t \in [0,T]$, and then a.e. in Q_T by du Bois-Reymond's lemma. □

In the rest of this section, we prove Proposition 4.3.14. For $\gamma \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$, we let $v_\varepsilon := \varepsilon B^{\mu}_{\varepsilon}[\gamma]$ be the unique smooth classical solution (see Theorem 4.3.1 (a) and Corollary 4.3.3 (a)) of

$$\varepsilon v_\varepsilon(x) - \mathcal{L}^{\mu}[v_\varepsilon](x) = \varepsilon\gamma(x) \qquad \text{for all} \qquad x \in \mathbb{R}^N. \tag{4.3.10}$$

We want to prove that there exists a sequence such that $v_{\varepsilon_n} = \varepsilon_n B^{\mu}_{\varepsilon_n}[\gamma] \to 0$ as $\varepsilon_n \to 0^+$ for every $x \in \mathbb{R}^N$ and every $\gamma \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$.

Lemma 4.3.15. *Assume* (A_μ) *and* $\gamma \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$. *Then there exists a sequence* $\{\varepsilon_n B^{\mu}_{\varepsilon_n}[\gamma]\}_{n\in\mathbb{N}}$ *that converges locally uniformly in* $\mathbb{R}^N$ *as* $\varepsilon_n \to 0^+$. *Moreover, the corresponding limit* v *is uniformly continuous,* $\lim_{|x|\to\infty} v = 0$ *and satisfies*

$$\mathcal{L}^{\mu}[v](x) = 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$

Lemma 4.3.16 (Barbălat)**.** *If* $\psi \in L^1(\mathbb{R}^N)$ *is uniformly continuous, then* $\lim\limits_{|x|\to\infty} \psi(x) = 0$.

For a proof, see e.g. Lemma 5.2 in [59] (take $G = \mathbb{R}^N$ and $B = \mathbb{R}$).

Proof of Lemma 4.3.15. We recall that $v_\varepsilon := \varepsilon B^{\mu}_{\varepsilon}[\gamma]$. By Theorem 4.3.1 (a),

$$\|D^{\alpha}v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \|D^{\alpha}\gamma\|_{L^\infty(\mathbb{R}^N)}$$

for each multiindex $\alpha \in \mathbb{N}^N$. So, then any sequence $\{v_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is equibounded and equilipschitz. By Arzelà-Ascoli's theorem, there exists a subsequence such that $v_{\varepsilon_n} \to v$ locally uniformly as $n \to \infty$. Since v_{ε_n} is uniformly continuous (the derivative of v_{ε_n} exists and is bounded) and by the local uniform convergence, for every $\eta > 0$ and $R > 0$ we can find some $n > 0$ such that $\max\{|v(x) - v_{\varepsilon_n}(x)| : |x| \leq R\} < \eta$. Thus, we have the following estimate for every $R > 0$ and $|x|, |y| \leq R$,

$$\begin{aligned}|v(x) - v(y)| &\leq |v(x) - v_{\varepsilon_n}(x)| + |v_{\varepsilon_n}(x) - v_{\varepsilon_n}(y)| + |v_{\varepsilon_n}(y) - v(y)| \\ &\leq 2\eta + \|D\gamma\|_{L^\infty(\mathbb{R}^N)}|x - y|\end{aligned}$$

As R is arbitrary, v is Lipschitz continuous with Lipschitz constant $\|D\gamma\|_{L^\infty(\mathbb{R}^N)}$, and thus, uniformly continuous. Furthermore, Fatou's lemma and Theorem 4.3.1 (b) give that $\|v\|_{L^1} \leq \liminf_{n\to\infty} \|v_{\varepsilon_n}\|_{L^1} \leq \|\gamma\|_{L^1}$. By Lemma 4.3.16, $\lim_{|x|\to\infty} v(x) = 0$.

Multiplying (4.3.10) by a test function, integrating over $\mathbb{R}^N$, and using self-adjointness (cf. Lemma 4.3.5) of $\mathcal{L}^\mu$ we get

$$\varepsilon_n \int_{\mathbb{R}^N} v_{\varepsilon_n}\psi \,\mathrm{d}x - \int_{\mathbb{R}^N} v_{\varepsilon_n}\mathcal{L}^\mu[\psi] \,\mathrm{d}x = \varepsilon_n \int_{\mathbb{R}^N} \gamma\psi \,\mathrm{d}x \qquad \text{for all} \qquad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N).$$

Since $\|v_{\varepsilon_n}\|_{L^\infty} \leq \|\gamma\|_{L^\infty}$ by Theorem 4.3.1 (c), we use Lebesgue's dominated convergence theorem to take the limit as $\varepsilon_n \to 0^+$, to find that

$$0 = \lim_{\varepsilon_n\to 0^+} \int_{\mathbb{R}^N} v_{\varepsilon_n}\mathcal{L}^\mu[\psi] \,\mathrm{d}x = \int_{\mathbb{R}^N} v\mathcal{L}^\mu[\psi] \,\mathrm{d}x \qquad \text{for all} \qquad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N),$$

which completes the proof. □

Lemma 4.3.17. *Assume* (A_μ) *and* $g \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. *Then there exists a sequence* $\{\varepsilon_n B^\mu_{\varepsilon_n}[g]\}_{n\in\mathbb{N}}$ *that converges in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *as* $\varepsilon_n \to 0^+$.

Proof. Note that $u_\varepsilon := \varepsilon B^\mu_\varepsilon[g]$ is the unique distributional solution (see Theorem 4.3.1 (b) and (c)) of the following elliptic problem

$$\varepsilon u_\varepsilon(x) - \mathcal{L}^\mu[u_\varepsilon](x) = \varepsilon g(x) \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$

By Theorem 4.3.1 (b) and (c) and the linearity of the above equation, for any $h \in \mathbb{R}^N$,

$$\|u_\varepsilon\|_{L^\infty} \leq \|g\|_{L^\infty}, \quad \|u_\varepsilon\|_{L^1} \leq \|g\|_{L^1} \quad \text{and} \quad \|u_\varepsilon(\cdot + h) - u_\varepsilon\|_{L^1} \leq \|g(\cdot + h) - g\|_{L^1}.$$

Now let $K \subset \mathbb{R}^N$ be any compact set, and define $w^K_\varepsilon(x) = u_\varepsilon(x)\mathbf{1}_K(x)$. The uniform in ε bound ensures that the family $M := \{w^K_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ is uniformly bounded in $L^1(\mathbb{R}^N)$. Moreover, by continuity of the L^1-translation, Theorem 4.3.1 (b) and (c), and

Lebesgue's dominated convergence theorem,

$$\begin{aligned}
&\|w_\varepsilon^K(\cdot+h) - w_\varepsilon^K\|_{L^1} \\
&\leq \| \left(u_\varepsilon(\cdot+h) - u_\varepsilon\right) \mathbf{1}_K(\cdot+h)\|_{L^1} + \|u_\varepsilon \left(\mathbf{1}_K(\cdot+h) - \mathbf{1}_K\right)\|_{L^1} \\
&\leq \|g(\cdot+h) - g\|_{L^1} + \|g\|_{L^\infty} \int_{\mathbb{R}^N} |\mathbf{1}_K(x+h) - \mathbf{1}_K(x)| \,\mathrm{d}x \to 0 \quad \text{as} \quad |h| \to 0.
\end{aligned}$$

Combining the above results, we see that M is relatively compact by Kolmogorov's compactness theorem (see e.g. [51, Theorem A.5]). Hence, there is a convergent subsequence in $L^1(K)$.

Now, cover $\mathbb{R}^N$ by a countable number of balls B_n. Then the above argument holds for $K := \overline{B}_n$ for every $n \in \mathbb{N}$. A diagonal argument then allows us to pick a subsequence which converges in $L^1(\overline{B}_n)$ for each n, and thus in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. □

Remark 4.3.18. By Theorem 4.3.1 (a) and Arzelà-Ascoli, we can have $D^\alpha v_\varepsilon \to w_\alpha$ locally uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0^+$ for all multiindex $\alpha \in \mathbb{N}^N$. However, because of the lack of uniqueness in $\mathcal{L}^\mu[v](x) = 0$, we do not know if $D^\alpha v = w_\alpha$. Hence, we are forced to work with distributional solutions of $\mathcal{L}^\mu[v](x) = 0$.

Lemma 4.3.19. *Assume* (A_μ)*,* $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$*, and* $\{\varepsilon_n B^\mu_{\varepsilon_n}[g]\}_{n\in\mathbb{N}}$ *converges in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$*. If* $\varepsilon_n B^\mu_{\varepsilon_n}[\gamma](x) \to 0$ *as* $\varepsilon_n \to 0^+$ *for every* $x \in \mathbb{R}^N$ *and every* $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$*, then* $\varepsilon_n B^\mu_{\varepsilon_n}[g] \to 0$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *as* $\varepsilon_n \to 0^+$*.*

Proof. By the self-adjointness given in Lemma 4.3.4, and the definitions $u_{\varepsilon_n} := \varepsilon_n B^\mu_\varepsilon[g]$, $v_{\varepsilon_n} := \varepsilon_n B^\mu_\varepsilon[\gamma]$, we have

$$\int_{\mathbb{R}^N} u_{\varepsilon_n}(x)\gamma(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} g(x) v_{\varepsilon_n}(x)\,\mathrm{d}x.$$

Since $\|v_{\varepsilon_n}\|_{L^\infty} \leq \|\gamma\|_{L^\infty}$ by Theorem 4.3.1 (c), $|g(x)v_{\varepsilon_n}(x)| \leq |g(x)|\|\gamma\|_{L^\infty}$. Then by the assumption and Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon_n\to 0^+} \int_{\mathbb{R}^N} u_{\varepsilon_n}(x)\gamma(x)\,\mathrm{d}x = 0 \qquad \text{for all} \qquad \gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N),$$

Hence $u_{\varepsilon_n} \to 0$ in $\mathcal{D}'(\mathbb{R}^N)$, and since the distributional and L^1_{loc} limits coincide (by uniqueness), it follows that $u_{\varepsilon_n} \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \to 0^+$. □

Proof of Proposition 4.3.14. Let $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$ be arbitrary, and recall the definitions $\varepsilon B^\mu_\varepsilon[\gamma] = v_\varepsilon$ and $\varepsilon B^\mu_\varepsilon[g] = u_\varepsilon$. Lemma 4.3.15 yields a subsequence such that $v_{\varepsilon_n} \to v$ locally uniformly as $\varepsilon_n \to 0^+$ with $v \in C_0(\mathbb{R}^N)$ and $\mathcal{L}^\mu[v](x) = 0$ in $D'(\mathbb{R}^N)$. Then, Theorem 4.3.9 ensures that $v(x) = 0$ for every $x \in \mathbb{R}^N$.

Hence, Lemma 4.3.17 and 4.3.19 give that $u_{\varepsilon_n} \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \to 0^+$. Finally, take a further subsequence (still denoted by ε_n) such that $u_{\varepsilon_n} \to 0$ a.e. in $\mathbb{R}^N$ as $\varepsilon_n \to 0^+$. □

4.4 Stability, existence and a priori results

In this section, we will start by showing the stability result stated in Section 4.2, and then we continue by showing existence and a priori results for (4.1.1). The latter part will follow by regularization and compactness from results in [33] for the case $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Proof of Theorem 4.2.6. Since u_n are distributional solutions of (4.1.1), we will take the limit as $n \to \infty$ to see that so are also u.

Assumption (iii) and the uniformly boundedness of $\|u_n\|_{L^\infty(Q_T)}$ gives for all $\psi \in C^\infty_{\mathrm{c}}(Q_T)$ that

$$\int_0^T \int_{\mathbb{R}^N} u_n \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^N} u \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \qquad \text{as} \qquad n \to \infty.$$

To prove convergence of the $\mathcal{L}^{\mu_n}$-term in the distributional formulation we proceed as follows

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \varphi_n(u_n)\mathcal{L}^{\mu_n}[\psi] - \varphi(u)\mathcal{L}[\psi] \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^N} \varphi_n(u_n)\big(\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]\big) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} \big(\varphi_n(u_n) - \varphi(u_n)\big)\mathcal{L}[\psi] \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^N} \big(\varphi(u_n) - \varphi(u)\big)\mathcal{L}[\psi] \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $\|u_n\|_{L^\infty(Q_T)}$ is uniformly bounded, $\varphi_n \to \varphi$ locally uniformly in $\mathbb{R}$ by assumption (ii), and $|\varphi_n(u_n)| \leq |\varphi_n(u_n) - \varphi(u_n)| + |\varphi(u_n)|$, we obtain for n sufficiently large

$$\|\varphi_n(u_n)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi(r)| : |r| \leq C\} + 1 =: C_\varphi \tag{4.4.1}$$

Then, using assumption (i), we get

$$\left| \int_0^T \int_{\mathbb{R}^N} \varphi_n(u_n) \left(\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]\right) \,\mathrm{d}x\,\mathrm{d}t \right| \leq C_\varphi \int_0^T \int_{\mathbb{R}^N} \left|\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]\right| \,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $n \to \infty$. By the uniformly boundedness of $\|u_n\|_{L^\infty(Q_T)}$, and since $\varphi_n \to \varphi$ locally uniformly in $\mathbb{R}$ by assumption (ii),

$$\|\varphi_n(u_n) - \varphi(u_n)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi_n(r) - \varphi(r)| : |r| \leq C\} \to 0 \quad \text{as} \quad n \to \infty.$$

Since we assume that $\mathcal{L}[\psi] \in L^1(Q_T)$,

$$\left| \int_0^T \int_{\mathbb{R}^N} \big(\varphi_n(u_n) - \varphi(u_n)\big)\mathcal{L}[\psi] \,\mathrm{d}x \,\mathrm{d}t \right| \leq \|\varphi_n(u_n) - \varphi(u_n)\|_{L^\infty} \|\mathcal{L}[\psi]\|_{L^1} \to 0$$

as $n \to \infty$. By assumption (iii) and (A_φ), $|\varphi(u_n) - \varphi(u)| \to 0$ a.e. in Q_T as $n \to \infty$, and $\|\varphi(u_n)\|_{L^\infty(Q_T)} \leq C$ for some C independent of n. Hence, $|\varphi(u_n) - \varphi(u)|$ is bounded by $2C$. Moreover, since $\mathcal{L}[\psi] \in L^1(Q_T)$, Lebesgue's dominated convergence theorem yields

$$\left| \int_0^T \int_{\mathbb{R}^N} (\varphi(u_n) - \varphi(u))\mathcal{L}[\psi] \,\mathrm{d}x \,\mathrm{d}t \right| \leq \int_0^T \int_{\mathbb{R}^N} |\varphi(u_n) - \varphi(u)||\mathcal{L}[\psi]| \,\mathrm{d}x \,\mathrm{d}t \to 0$$

as $n \to \infty$. The proof is complete. $\square$

Let us turn our attention to proving the other main results in this section.

Theorem 4.4.1. *Assume* (A_φ), (A_μ), $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

(a) There exists a unique entropy solution $u \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^N))$ *of* (4.1.1).

(b) If $u, \hat{u}$ *are entropy solutions of* (4.1.1) *with initial data* $u_0, \hat{u}_0$ *respectively, then for all* $t \in [0,T]$

$$\|u(\cdot,t) - \hat{u}(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0 - \hat{u}_0\|_{L^1(\mathbb{R}^N)}.$$

(c) If u *is a entropy solution of* (4.1.1) *with initial data* u_0*, then for all* $t \in [0,T]$

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} \qquad \text{and} \qquad \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Entropy solutions are defined in Definition 2.1 in [33], and the result holds by Theorem 5.5 in [33] and Theorem 5.2 in [31].

In what follows, we let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and define

$$\varphi_\eta(x) := \varphi * \omega_\eta(x) \quad \text{where} \quad \omega_\eta \quad \text{is given by (4.1.7) with } N = 1. \tag{4.4.2}$$

Hence $\varphi_\eta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \subset C(\mathbb{R})$, it is nondecreasing by (A_φ), and $\varphi_\eta \to \varphi$ locally uniformly in $\mathbb{R}$. Let u_η be the entropy solution of (4.1.1) with φ_η replacing φ. Since entropy solutions are distributional solutions (cf. Theorem 2.5 ii) and Section 5 in [31]),

$$\int_0^T \int_{\mathbb{R}^N} u_\eta \partial_t \psi + \varphi_\eta(u_\eta)\mathcal{L}^\mu[\psi] \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0 \psi|_{t=0} \,\mathrm{d}x = 0 \quad \forall \psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N \times [0,T)). \tag{4.4.3}$$

Going to the limit as $\eta \to 0^+$ in (4.4.3), we will prove the existence and the a priori results given in Theorems 4.2.8 and 4.2.9.

Remark 4.4.2. We will prove that the L^1-contraction holds for limits of the functions $\{u_\eta\}_{\eta>0}$. As a consequence of uniqueness (Corollary 4.2.4), this result then holds for all $L^\infty \cap L^1$-distributional solutions of (4.1.1).

Before these results can be proved, we need an auxiliary lemma.

Lemma 4.4.3. *Assume* (A_μ), $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, φ_η *satisfy* (A_φ) *for all* $\eta > 0$, *and* $\varphi_\eta \to \varphi$ *locally uniformly as* $\eta \to 0^+$. *If* u_η *solves* (4.4.3) *and satisfies Theorem 4.4.1 (b) and (c), then there exists a subsequence* $\{u_{\eta_n}\}_{n\in\mathbb{N}}$ *and a* $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *such that as* $\eta_n \to 0^+$

$$u_{\eta_n} \to u \qquad \textit{in} \qquad C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Moreover, for all $t \in [0,T]$

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} \qquad \textit{and} \qquad \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. We will use Kolmogorov's compactness theorem in the form of Theorem A.8 in [51]. Let $K \subset \mathbb{R}^N$ be any compact set.

Step 1: u_η is bounded independently of η in Q_T by Theorem 4.4.1 (c).

Step 2: Since (4.1.1) is translation invariant, $v(x,t) = u_\eta(x+h,t)$ solves (4.4.3) with initial data $v_0(x) = u_0(x+h)$ for every $h \in \mathbb{R}^N$. Let $\gamma \in \mathbb{R}^N$. By Theorem 4.4.1 (b) and since translations are continuous in L^1,

$$\begin{aligned}
&\sup_{|h|\leq|\gamma|} \int_K |u_\eta(x+h,t) - u_\eta(x,t)| \,\mathrm{d}x \leq \sup_{|h|\leq|\gamma|} \int_{\mathbb{R}^N} |u_\eta(x+h,t) - u_\eta(x,t)| \,\mathrm{d}x \\
&\leq \sup_{|h|\leq|\gamma|} \int_{\mathbb{R}^N} |u_0(x+h) - u_0(x)| \,\mathrm{d}x \leq \max_{|h|\leq|\gamma|} \tilde{\lambda}_{u_0}(|h|) =: \lambda_{u_0}(|\gamma|)
\end{aligned}$$

for some moduli of continuity $\tilde{\lambda}_{u_0}, \lambda_{u_0}$.

Step 3: Let ω_δ be defined by (4.1.7) and let $\Theta \in C_\mathrm{c}^\infty((0,T))$. For any $x \in \mathbb{R}^N$ take $\psi(y,t) = \Theta(t)\omega_\delta(x-y)$ as a test function in (4.4.3) to find that

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^N} u_\eta(y,t)\omega_\delta(x-y)\Theta'(t) + \varphi_\eta(u_\eta(y,t))\mathcal{L}^\mu[\omega_\delta](x-y)\Theta(t) \,\mathrm{d}y \,\mathrm{d}t \\
&= \int_0^T \big(u_\eta(\cdot,t) * \omega_\delta\big)(x)\Theta'(t) + \big(\varphi_\eta(u_\eta(\cdot,t)) * \mathcal{L}^\mu[\omega_\delta]\big)(x)\Theta(t) \,\mathrm{d}t.
\end{aligned} \tag{4.4.4}$$

For $\rho_\delta(t)$ defined by (4.1.8), we choose

$$\Theta(t) := \Theta_{\tilde{\delta}}(t) = \int_{-\infty}^t \rho_{\tilde{\delta}}(\tau - t_1) - \rho_{\tilde{\delta}}(\tau - t_2) \,\mathrm{d}\tau,$$

where $0 < t_1 < t_2 < T$. For $\tilde{\delta} > 0$ small enough, $\Theta_{\tilde{\delta}}(t)$ is supported in $[0, T]$ and is a smooth approximation to a square pulse which is one in $[t_1, t_2]$ and zero otherwise. By (4.4.4),

$$\int_0^T \rho_{\tilde{\delta}}(t - t_2)\big(u_\eta(\cdot, t) * \omega_\delta\big)(x)\,\mathrm{d}t = \int_0^T \rho_{\tilde{\delta}}(t - t_1)\big(u_\eta(\cdot, t) * \omega_\delta\big)(x)\,\mathrm{d}t + \int_0^T \Theta_{\tilde{\delta}}(t)\big(\varphi_\eta(u_\eta(\cdot, t)) * \mathcal{L}^\mu[\omega_\delta]\big)(x)\,\mathrm{d}t.$$

Let $u_\eta^\delta(x, t) := u_\eta(\cdot, t) * \omega_\delta(x)$. By Theorem 4.4.1 (c) and the properties of mollifiers, we send $\tilde{\delta} \to 0^+$ in the previous equality to obtain the following pointwise identity,

$$u_\eta^\delta(x, t_2) - u_\eta^\delta(x, t_1) = \int_{t_1}^{t_2} \big(\varphi_\eta(u_\eta(\cdot, t)) * \mathcal{L}^\mu[\omega_\delta]\big)(x)\,\mathrm{d}t. \tag{4.4.5}$$

Now, we need to estimate the integral involving the mollified version of u_η. Let $t, s \in [0, T]$ and take $\delta < \min\{t, s\}$. Use (4.4.5) to find that

$$\begin{aligned}\int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)|\,\mathrm{d}x &\leq \int_K \int_s^t \left|\big(\varphi_\eta(u_\eta(\cdot, \tau)) * \mathcal{L}^\mu[\omega_\delta]\big)(x)\right|\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_s^t \int_K \int_{\mathbb{R}^N} |\varphi_\eta(u_\eta(x - y, \tau))|\,|\mathcal{L}^\mu[\omega_\delta](y)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq \|\varphi_\eta(u_\eta)\|_{L^\infty(Q_T)} \|\mathcal{L}^\mu[\omega_\delta]\|_{L^1(\mathbb{R}^N)} |K||t - s|,\end{aligned}$$

where $|K|$ denotes the Lebesgue measure of the compact set K. As in the proof of Theorem 4.2.6 (see (4.4.1)), we obtain for η sufficiently small

$$\|\varphi_\eta(u_\eta)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi(r)| : |r| \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\} + 1.$$

Moreover, Lemma 4.3.5 (b) yields

$$\|\mathcal{L}^\mu[\omega_\delta]\|_{L^1(\mathbb{R}^N)} \leq (\delta^{-2} + 1) \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z).$$

Hence, taking $\delta^2 := |t - s|^{\frac{2}{3}}$ we see that

$$\int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)|\,\mathrm{d}x \leq \tilde{C}_{K,\varphi,u_0,\mu} \left(|t - s|^{\frac{1}{3}} + |t - s|\right), \tag{4.4.6}$$

where

$$\tilde{C}_{K,\varphi,u_0,\mu} = |K| \left(\sup\{|\varphi(r)| : |r| \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\} + 1\right) \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z).$$

By the triangle inequality and Theorem 4.4.1 (b),

$$\begin{aligned}
&\int_K |u_\eta(x,t)-u_\eta(x,s)|\,\mathrm{d}x\\
&\leq \int_K |u_\eta(x,t)-u_\eta^\delta(x,t)|\,\mathrm{d}x+\int_K |u_\eta^\delta(x,t)-u_\eta^\delta(x,s)|\,\mathrm{d}x\\
&\quad+\int_K |u_\eta^\delta(x,s)-u_\eta(x,s)|\,\mathrm{d}x\\
&\leq \sup_{|\sigma|\leq\delta}\|u_\eta(\cdot,t)-u_\eta(\cdot+\sigma,t)\|_{L^1(\mathbb{R}^N)}+\int_K |u_\eta^\delta(x,t)-u_\eta^\delta(x,s)|\,\mathrm{d}x\\
&\quad+\sup_{|\sigma|\leq\delta}\|u_\eta(\cdot,s)-u_\eta(\cdot+\sigma,s)\|_{L^1(\mathbb{R}^N)}\\
&\leq 2\sup_{|\sigma|\leq\delta}\|u_0-u_0(\cdot+\sigma)\|_{L^1(\mathbb{R}^N)}+\int_K |u_\eta^\delta(x,t)-u_\eta^\delta(x,s)|\,\mathrm{d}x\\
&\leq 2\max_{|\sigma|\leq\delta}\tilde{\lambda}_{u_0}(\delta)+\int_K |u_\eta^\delta(x,t)-u_\eta^\delta(x,s)|\,\mathrm{d}x,
\end{aligned}$$

where $\tilde{\lambda}_{u_0}$ is defined in Step 2 . Hence, by (4.4.6)

$$\begin{aligned}
\int_K |u_\eta(x,t)-u_\eta(x,s)|\,\mathrm{d}x &\leq \lambda_{u_0}\left(|t-s|^{\frac{1}{3}}\right)+\tilde{C}_{K,\varphi,u_0,\mu}\left(|t-s|^{\frac{1}{3}}+|t-s|\right)\\
&=: \Lambda_{K,\varphi,u_0,\mu}(|t-s|)
\end{aligned}$$

for some moduli of continuity λ_{u_0} and $\Lambda_{K,\varphi,u_0,\mu}$.

Step 4: The assumptions of Theorem A.8 in [51] hold by Steps 1–3, so we conclude that there is a subsequence $\{u_{\eta_n}\}_{n\in\mathbb{N}}$ such that

$$u_{\eta_n}\to u \qquad \text{in} \qquad C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$$

as $\eta_n\to 0^+$. Finally, u inherits the properties of u_η given in Theorem 4.4.1 (c) by Fatou's lemma, and the fact that the limit of a uniformly bounded sequence which converges a.e. is also bounded. □

Remark 4.4.4. If $\mathcal{L}^\mu$ was not fixed in the above result, but rather $\mu=\mu_n$ (with μ_n satisfying (A_μ)), then the result still holds and the proof is the same provided we also assume that for some $a>0$ there exists a function $f\in L^\infty_{\mathrm{loc}}((0,a))$ such that

$$\|\mathcal{L}^{\mu_n}[\omega_\delta]\|_{L^1(\mathbb{R}^N)}\leq f(\delta) \qquad \text{for every} \qquad \delta\in(0,a),$$

where ω_δ is defined by (4.1.7). Observe that the above inequality follows from the assumption $\sup_n\int_{|z|>0}\min\{|z|^2,1\}\,\mathrm{d}\mu_n(z)<\infty$ in Theorem 4.2.11.

Now, the proofs of the existence and the a priori results follow.

Proof of Theorem 4.2.8. Let u_{η_n} be the solutions of (4.4.2) (cf. Theorem 4.4.1), $u \in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N))$ the function provided by Lemma 4.4.3, and define $\mathcal{L}^{\mu_n} := \mathcal{L}^\mu$ (that is, $\mathcal{L}^{\mu_n} := \mathcal{L}^\mu = \mathcal{L}$), $\varphi_n := \varphi_{\eta_n}$, and $u_n := u_{\eta_n}$. Then assumptions (i), (ii), and (iii) in Theorem 4.2.6 are satisfied by the n-independence of $\mathcal{L}^\mu$, (4.4.2), and Lemma 4.4.3. Moreover, $\sup_n \|u_n\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} < \infty$ by Theorem 4.4.1 (c). Hence, by Theorem 4.2.6, u satisfies (4.1.1) in the sense of distributions; cf. Lemma 4.2.20 and Definition 4.2.2. Moreover, we have that $u - u_0 \in L^1(Q_T)$. So, u is in fact a distributional solution of (4.1.1) in the sense of Definition 4.2.2, and it is unique by Corollary 4.2.4.

Thus, any subsequence has the same limit, and hence, the whole sequence $\{u_\eta\}_{\eta>0}$ converges since it is bounded by Theorem 4.4.1 (c). □

Proof of Theorem 4.2.9. (a) Let u_η be the entropy solution of (4.4.2) (cf. Theorem 4.4.1). Using the semi entropy-entropy flux pairs

$$(u_\eta - k)^\pm \qquad \text{and} \qquad \pm\,\text{sign}^\pm(u_\eta - k)(f(u_\eta) - f(k)) \qquad \text{for all } k \in \mathbb{R},$$

and the corresponding definitions for entropy solutions in stead of the Kružkov entropy-entropy flux pairs in [31], we obtain

$$\int_{\mathbb{R}^N} (u_\eta(x,t) - \hat{u}_\eta(x,t))^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ \,\mathrm{d}x$$

for $u_\eta, \hat{u}_\eta \in L^\infty(Q_T)\cap L^1(Q_T)\cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N))$ with initial data $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$. See [45] for the result and a proof.

By Lemma 4.4.3, we can take subsequences such that $u_{\eta_n}, \hat{u}_{\eta_n} \to u, \hat{u}$ a.e. in Q_T as $\eta_n \to 0^+$. Thus, Fatou's lemma yields the result.

(b) By the contraction estimate obtained in part (a) and $u_0 \leq \hat{u}_0$ a.e. in $\mathbb{R}^N$, for all $t \in (0,T)$, $\int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))^+ \,\mathrm{d}x \leq 0$. Hence, $(u-\hat{u})^+ = 0$ and $u \leq \hat{u}$ a.e. in Q_T.

(c) Follows by Lemma 4.4.3.

(d) Follows by Lemma 4.4.3.

(e) Using the triangle inequality, and taking u, u_{η_n} as in Lemma 4.4.3, we obtain by Step 3 in the proof of that lemma that for all $t, s \in [0, T]$ and any compact set $K \subset \mathbb{R}^N$

$$
\begin{aligned}
&\|u(\cdot,t)-u(\cdot,s)\|_{L^1(K)}\\
&\le \|u(\cdot,t)-u_{\eta_n}(\cdot,t)\|_{L^1(K)} + \|u_{\eta_n}(\cdot,t)-u_{\eta_n}(\cdot,s)\|_{L^1(K)} + \|u_{\eta_n}(\cdot,s)-u(\cdot,s)\|_{L^1(K)}\\
&\le 2\|u(\cdot,t)-u_{\eta_n}(\cdot,t)\|_{C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))} + \Lambda_{K,\varphi,u_0,\mu}(|t-s|)
\end{aligned}
$$

for the modulus of continuity $\Lambda_{K,\varphi,u_0,\mu}$ (see the above mentioned proof). Since $u_{\eta_n} \to u$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ by Lemma 4.4.3, the proof is complete.

(f) Consider a standard cut-off function $0 \le \mathcal{X} \in C_c^\infty(\mathbb{R}^N)$ such that $\mathcal{X}(x) = 1$ for $|x| \le 1$ and $\mathcal{X}(x) = 0$ for $|x| \ge 2$. We will write $\mathcal{X}_R(x) = \mathcal{X}(\frac{x}{R})$ for $R > 0$. Following the proof of Lemma 4.3.10 (b), with θ_a as defined there, we can take $\psi(x,t) = \mathcal{X}_R(x)\theta_a(t)$ for any $R > 0$ as a test function in Definition 4.2.2 (cf. Lemma 4.2.20). Hence

$$
\begin{aligned}
\frac{1}{a}\int_{s-a}^{s}\int_{\mathbb{R}^N} u(x,t)\mathcal{X}_R(x)\,\mathrm{d}x\,\mathrm{d}t = &\int_0^s \theta_a(t)\int_{\mathbb{R}^N}\varphi(u(x,t))\mathcal{L}^\mu[\mathcal{X}_R](x)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\mathbb{R}^N} u_0(x)\mathcal{X}_R(x)\,\mathrm{d}x.
\end{aligned}
$$

Since $\mathcal{X}_R$ is compactly supported and $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$, we can pass to the limit as $a \to 0^+$ in the first integral to get $\int_{\mathbb{R}^N} u(x,s)\mathcal{X}_R(x)\,\mathrm{d}x$. For the second integral, we know that $\varphi(u) \in L^\infty(Q_T)$, $\mathcal{L}^\mu[\mathcal{X}_R] \in L^1(\mathbb{R}^N)$ and $\theta_a \to \mathbf{1}_{[0,s)}$ pointwise a.e. as $a \to 0^+$, and thus, it converges as $a \to 0^+$ to $\int_0^s\int_{\mathbb{R}^N}\varphi(u(x,t))\mathcal{L}^\mu[\mathcal{X}_R](x)\,\mathrm{d}x\,\mathrm{d}t$ by Lebesgue's dominated convergence theorem. In this way, we get

$$
\int_{\mathbb{R}^N} u(x,s)\mathcal{X}_R(x)\,\mathrm{d}x = \int_0^s\int_{\mathbb{R}^N}\varphi(u(x,t))\mathcal{L}^\mu[\mathcal{X}_R](x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\mathcal{X}_R(x)\,\mathrm{d}x.
$$

The function $\mathcal{X}_R$ converges pointwise as $R \to \infty$ to 1, and it is also bounded by 1. Then, since $u(\cdot,s), u_0 \in L^1(\mathbb{R}^N)$, Lebesgue's dominated convergence theorem allows us to pass to the limit as $R \to \infty$ in the first and the last integrals to get $\int_{\mathbb{R}^N} u(x,s)\,\mathrm{d}x$ and $\int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$, respectively, for all $s \in (0,T)$. Consider the nonsingular part of the Lévy operator, i.e., $\int_{|z|>1}\mathcal{X}_R(x+z)-\mathcal{X}_R(x)\,\mathrm{d}\mu(z)$ which is bounded by $2\mu(\{z \in \mathbb{R}^N : |z| > 1\})$ for every $x \in \mathbb{R}^N$. Since $\mathcal{X}_R(y) \to 1$ pointwise as $R \to \infty$ for all $y \in \mathbb{R}^N$, Lebesgue's dominated convergence theorem shows the pointwise convergence to 0 of the nonsingular part. For the singular part, Lemma 4.3.5 (b) gives

$$
\left|\int_{0<|z|\le 1}\mathcal{X}_R(x+z)-\mathcal{X}_R(x)\,\mathrm{d}\mu(z)\right| \le \frac{1}{R^2}\|D^2\mathcal{X}\|_{L^\infty(\mathbb{R}^N)}\int_{|z|\le 1}|z|^2\,\mathrm{d}\mu(z)
$$

which also goes to 0 as $R \to \infty$. Moreover, by the assumption $|\varphi(r)| \leq L_\delta |r|$ for $|r| \leq \delta$,

$$\|\varphi(u(x,t))\|_{L^1(Q_T)} \leq \int_0^T \int_{|u|\leq\delta} L_\delta |u(x,t)| \,\mathrm{d}x \,\mathrm{d}t + \|\varphi(u)\|_{L^\infty(Q_T)} \int_0^T \int_{|u|>\delta} \mathrm{d}x \,\mathrm{d}t.$$

Since $u \in L^1(Q_T)$, both terms on the right-hand side of the estimate above are finite. Then by Lebesgue's dominated convergence theorem,

$$\left| \int_0^s \int_{\mathbb{R}^N} \varphi(u(x,t)) \mathcal{L}^\mu[\mathcal{X}_R](x) \,\mathrm{d}x \,\mathrm{d}t \right| \to 0 \qquad \text{as} \qquad R \to \infty.$$

The proof is complete. $\square$

4.5 Applications of stability

This section focuses on proving the results stated in Sections 4.2.1 and 4.2.2.

4.5.1 Compactness, local limits and continuous dependence

Proof of Theorem 4.2.11. (a) Note that the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ satisfy the hypothesis of Theorem 4.2.9. By the assumptions, Remark 4.4.4, and Lemma 4.4.3, the result follows.

(b) This is a consequence of the stability given in Theorem 4.2.6. For the initial condition, note that by the assumption $\sup_n \|u_{0,n}\|_{L^\infty(\mathbb{R}^N)} < \infty$ and Fatou's lemma, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, and the convergence of $\int_{\mathbb{R}^N} u_{0,n}(x)\psi(x,0)\,\mathrm{d}x$ follows by the L^1_{loc}-convergence of $\{u_{0,n}\}_{n\in\mathbb{N}}$. $\square$

Lemma 4.5.1. *Assume* (A_μ), $s \in (0,2)$, $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, *and* $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. *Then*

$$\lim_{s\to 2^-} \left\| -(-\Delta)^{\frac{s}{2}}\psi - \Delta\psi \right\|_{L^1(\mathbb{R}^N)} = 0.$$

Proof. The fractional Laplacian has a representation in the form (4.1.3) with measure

$$\mathrm{d}\mu = c_{N,s} \frac{\mathrm{d}z}{|z|^{N+s}} \qquad \text{for} \qquad c_{N,s} = \left(N \int_{\mathbb{R}^N} \frac{1-\cos(z_1)}{|z|^{N+s}} \,\mathrm{d}z \right)^{-1},$$

where

$$\lim_{s\to 2^-} c_{N,s} = 0, \tag{4.5.1}$$

see e.g. Proposition 4.1 in [42]. Hence

$$-(-\Delta)^{\frac{s}{2}}\psi(x) = c_{N,s}\int_{|z|\leq 1}\frac{\psi(x+z)-\psi(x)-z\cdot D\psi(x)}{|z|^{N+s}}\,\mathrm{d}z + c_{N,s}\int_{|z|>1}\frac{\psi(x+z)-\psi(x)}{|z|^{N+s}}\,\mathrm{d}z,$$

where the last term goes to zero in $L^1(\mathbb{R}^N)$ as $s\to 2^-$ since it is bounded in $L^1(\mathbb{R}^N)$ by $c_{N,s}2\|\psi\|_{L^1(\mathbb{R}^N)}\int_{|z|>1}|z|^{-N-1}\,\mathrm{d}z$ for $s\geq 1$.

The explicit form of $c_{N,s}$ given in (4.5.1) yields

$$\Delta\psi(x) = \Delta\psi(x)c_{N,s}N\int_{|z|\leq 1}\frac{1-\cos(z_1)}{|z|^{N+s}}\,\mathrm{d}z + \Delta\psi(x)c_{N,s}N\int_{|z|>1}\frac{1-\cos(z_1)}{|z|^{N+s}}\,\mathrm{d}z.$$

Again, the last term goes to zero in $L^1(\mathbb{R}^N)$ as $s\to 2^-$ since $|1-\cos(z_1)|\leq 2$ and then it is bounded in $L^1(\mathbb{R}^N)$ by $c_{N,s}2N\int_{|z|>1}|z|^{-N-1}\,\mathrm{d}z$ for $s\geq 1$. Using Taylor's theorem, we see that $1-\cos(z_1)=\frac{1}{2}z_1^2-\frac{1}{24}z_1^4\cos(\xi)$ for some $\xi\in[0,z_1]$. Hence,

$$\int_{|z|\leq 1}\frac{1-\cos(z_1)}{|z|^{N+s}}\,\mathrm{d}z = \frac{1}{2}\int_{|z|\leq 1}\frac{z_1^2}{|z|^{N+s}}\,\mathrm{d}z - \frac{1}{24}\cos(\xi)\int_{|z|\leq 1}\frac{z_1^4}{|z|^{N+s}}\,\mathrm{d}z, \tag{4.5.2}$$

and the following estimate holds:

$$\left\|\frac{N}{24}\Delta\psi(x)c_{N,s}\int_{|z|\leq 1}\frac{\cos(\xi)z_1^4}{|z|^{N+s}}\,\mathrm{d}z\right\|_{L^1(\mathbb{R}^N)} \leq \frac{N}{24}c_{N,s}\|\Delta\psi\|_{L^1(\mathbb{R}^N)}\int_{|z|\leq 1}\frac{1}{|z|^{N-2}}\,\mathrm{d}z$$

which goes to zero since $\int_{|z|\leq 1}\frac{1}{|z|^{N-2}}\,\mathrm{d}z<\infty$ and (4.5.1) hold.

To estimate the remaining term in (4.5.2), note that for all $r>0$,

$$N\Delta\psi(x)\int_{|z|\leq r}z_1^2\,\mathrm{d}z = \Delta\psi(x)\int_{|z|\leq r}|z|^2\,\mathrm{d}z = \int_{|z|\leq r}D^2\psi(x)z\cdot z\,\mathrm{d}z,$$

and then

$$\frac{1}{2}c_{N,s}N\Delta\psi(x)\int_{|z|\leq 1}\frac{z_1^2}{|z|^{N+s}}\,\mathrm{d}z = c_{N,s}\int_{|z|\leq 1}\frac{\frac{1}{2}D^2\psi(x)z\cdot z}{|z|^{N+s}}\,\mathrm{d}z.$$

We combine the all above estimates to get

$$\begin{aligned}
&\lim_{s\to 2^-}\|-(-\Delta)^{\frac{s}{2}}\psi(x)-\Delta\psi(x)\|_{L^1(\mathbb{R}^N)}\\
&=\lim_{s\to 2^-}c_{N,s}\left\|\int_{|z|\leq 1}\frac{\psi(x+z)-\psi(x)-z\cdot D\psi(x)-\frac{1}{2}D^2\psi(x)z\cdot z}{|z|^{N+s}}\,\mathrm{d}z\right\|_{L^1(\mathbb{R}^N)}+0\\
&\leq \lim_{s\to 2^-}c_{N,s}\frac{1}{6}\|D^3\psi\|_{L^1(\mathbb{R}^N)}\int_{|z|\leq 1}\frac{|z|^3}{|z|^{N+s}}\,\mathrm{d}z,
\end{aligned}$$

where the last inequality follows from Taylor's and Fubini's theorems. Since the z-integral is bounded by $\int_{|z|\leq 1} \frac{1}{|z|^{N-1}} \,\mathrm{d}z < \infty$ and (4.5.1) hold, the limit is zero and the proof is complete. □

Proof of Corollary 4.2.12. (a) We will use Theorem 4.2.11 and Remark 4.4.4 to prove the result, and now we verify the assumptions. By Lemma 4.5.1, $-(-\Delta)^{\frac{s}{2}}\psi \to \Delta\psi$ in $L^1(\mathbb{R}^N)$ as $s \to 2^-$ for all $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. Moreover, by Lemma 4.3.5 (b), properties of mollifiers, $\lim_{s\to 2^-} \int_{|z|\leq 1} |z|^2 \frac{c_{N,s}\,\mathrm{d}z}{|z|^{N+s}} = 1$, and $\lim_{s\to 2^-} \int_{|z|>1} \frac{c_{N,s}\,\mathrm{d}z}{|z|^{N+s}} = 0$ (see previous proof),

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}\omega_\delta\|_{L^1} &= \frac{1}{2}\|D^2\omega_\delta\|_{L^1}\int_{|z|\leq 1}|z|^2\frac{c_{N,s}\,\mathrm{d}z}{|z|^{N+s}} + 2\|\omega_\delta\|_{L^1}\int_{|z|>1}\frac{c_{N,s}\,\mathrm{d}z}{|z|^{N+s}}\\ &\leq C\Big(1+\frac{1}{\delta^2}\|D^2\omega\|_{L^1}\Big)\end{aligned}$$

for s close to 2. Hence, since also φ is fixed (independently of s), we may use Theorem 4.2.11 and Remark 4.4.4 to get a subsequence $\{u_{s_j}\}_{j\in\mathbb{N}}$ and a $u \in C([0,T]; L^1_\mathrm{loc}(\mathbb{R}^N))$ such that $u_{s_j} \to u$ in $C([0,T]; L^1_\mathrm{loc}(\mathbb{R}^N))$ as $j \to \infty$. Finally, the uniqueness for the limit (equation) [19], and the boundedness of the sequence $\{u_s\}_{s\in(0,2)}$ (Theorem 4.2.9 (d)), ensures that the whole sequence converges.

(b) Since $(-\Delta)^{\frac{s_n}{2}}\psi \to (-\Delta)^{\frac{\bar{s}}{2}}\psi$ in $L^1(\mathbb{R}^N)$ as $n\to\infty$ (a similar argument as in Lemma 4.5.1), $\varphi_{m_n}(r) = r^{m_n} \to \varphi_{\bar{m}}(r) = r^{\bar{m}}$ locally uniformly as $n \to \infty$, and $\|(-\Delta)^{\frac{s_n}{2}}\omega_\delta\|_{L^1} \leq C(1+\delta^{-2})$ by the proof of part (a), convergence for a subsequence follows by Theorem 4.2.11. Moreover, the convergence of the whole sequence follows from uniquenes of the limit (Corollary 4.2.4) and boundedness of the sequence (Theorem 4.2.9 (d)). □

4.5.2 Numerical approximation, convergence and existence

We start by showing that a standard finite difference approximations of the Laplacian can be written in the from (4.1.3) and that convergence of the resulting scheme then follows from our theory.

Example 4.2. *Let $e_i \in \mathbb{R}^n$ for $i = 1, ..., N$ be points with i-th component 1 and the other components 0. Using δ-measures and $h > 0$, we define*

$$\mu_h = \sum_{i=1}^{N} \frac{\delta_{he_i} + \delta_{-he_i}}{h^2}.$$

It is clear that μ_h *is a measure satisfying* (A_μ) *for every* $h > 0$. *Moreover,*

$$\mathcal{L}^{\mu_h}[v](x) := \int_{\mathbb{R}^N} v(x+z) - v(x)\; d\mu_h(z) = \sum_{i=1}^{N} \frac{v(x+he_i) + v(x-he_i) - 2v(x)}{h^2}.$$

With $\mu = \mu_h$, *problem* (4.2.5) *can be reformulated as*

$$\partial_t u_h(x,t) - \sum_{i=1}^{N} \frac{\varphi(u_h(x+he_i,t)) + \varphi(u_h(x-he_i,t)) - 2\varphi(u_h(x,t))}{h^2} = 0 \tag{4.5.3}$$

in $\mathcal{D}'(Q_T)$.

For $\psi \in C_c^\infty(\mathbb{R}^N)$, *an application of Taylor's theorem reveals that there is a* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} |\mathcal{L}^{\mu_h}[\psi](x) - \Delta\psi(x)|\,\mathrm{d}x \leq h^2 C\|D^4\psi\|_{L^1(\mathbb{R}^N)} \to 0 \quad \textit{as} \quad h \to 0^+.$$

Moreover, for h *small enough,*

$$\sup_h \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu_h(z) = \sup_h \sum_{i=1}^{N} \frac{|he_i|^2 + |-he_i|^2}{h^2} = 2N$$

Then by Theorem 4.2.11, there existis a subsequence $\{u_{h_j}\}_{j\in\mathbb{N}}$ *of solutions of* (4.5.3), *and a* $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *such that* $u_{h_j} \to u$ *in* $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *as* $j \to \infty$. *Moreover, the limit* u *satisfies equation* (4.1.5):

$$\partial_t u - \Delta\varphi(u) = 0 \qquad \textit{in} \qquad \mathcal{D}'(Q_T).$$

In fact, as in the proof of Corollary 4.2.12, the whole sequence $\{u_h\}_{h>0}$ *converges.*

We can proceed as in this example to get convergence for a more general class of second order local operators.

Lemma 4.5.2. *Assume* $h > 0$, $P \in \mathbb{N}$, $\sigma = (\sigma_1, ..., \sigma_P)$, $\sigma_i \in \mathbb{R}^N$ *for* $i = 1, ..., P$, L_h^σ *is defined by* (4.2.9), *and* $\psi \in C_c^\infty(\mathbb{R}^N)$. *Then*

$$L_h^\sigma[\psi](x) = \int_{|z|>0} \psi(x+z) - \psi(x)\,\mathrm{d}\mu_h(z) =: \mathcal{L}^{\mu_{h,\sigma}}[\psi](x),$$

where the measure $\mu_{h,\sigma} = \frac{1}{h^2}\sum_{i=1}^{P}(\delta_{h\sigma_i} + \delta_{-h\sigma_i})$. *Moreover,* $\mu_{h,\sigma}$ *satisfies* (A_μ),

$$\sup_h \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu_{h,\sigma}(z) < \infty,$$

and

$$\|L^{\mu_{h,\sigma}}[\psi] - \mathrm{tr}[\sigma\sigma^T D^2\psi]\|_{L^1} \to 0 \qquad \textit{as} \qquad h \to 0^+.$$

Proof. By an elementary identity and Talyor's theorem,

$$\begin{aligned}\operatorname{tr}[\sigma\sigma^T D^2\psi(x)] &= \sum_{i=1}^{P}(\sigma_i \cdot D)^2\psi(x)\\ &= \sum_{i=1}^{P}\frac{v(x+h\sigma_i)+v(x-h\sigma_i)-2v(x)}{h^2} + h^2\sum_{i=1}^{N}\sum_{|\beta|=4}\frac{1}{\beta!}\sigma_i^\beta D^\beta\psi(\xi_i)\end{aligned}$$

Here we use standard multiindex notation, with multiindex $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}^N$, to account for the 4-th order derivatives. Since the first term of the last line is $L_h^\sigma[\psi](x)$, the rest of the proof follows along the arguments of Example 4.2. □

We aim to consider the general operator $\mathcal{L}^\mu$ defined in (4.1.3). In order to use our stability result, we would like to prove that the operator $\mathcal{L}_h^\mu$ defined in (4.2.10) is a particular case of the operators studied in this chapter. The following result ensures this fact.

Lemma 4.5.3. *Assume* (A_μ)*,* $h>0$*,* $\mathcal{L}_h^\mu$ *is defined in* (4.2.10)*, and* $\psi\in C_c^\infty(\mathbb{R}^N)$*. Then*

$$\mathcal{L}_h^\mu[\psi](x) = \int_{|z|>0}\psi(x+z)-\psi(x)\,\mathrm{d}\nu_h(z) =: \mathcal{L}^{\nu_h}[\psi](x)$$

where the measure $\nu_h=\sum_{\alpha\neq 0}\mu\left(z_\alpha+R_h\right)\delta_{z_\alpha}$*. Moreover,* ν_h *satisfies* (A_μ) *and*

$$\sup_h\int_{|z|>0}\min\{|z|^2,1\}\,\mathrm{d}\nu_h(z)<\infty.$$

Proof. By the definition of δ_{z_α}, it immetiatly follows that $\mathcal{L}_h^\mu=\mathcal{L}^{\nu_h}$. It remains to show that ν_h satisfies (A_μ). For $h<1/\sqrt{N}$,

$$\int_{|z|>1}\mathrm{d}\nu_h(z) = \sum_{|z_\alpha|>1}\mu\left(z_\alpha+R_h\right)\le\mu\left(\left\{|z|>1-\sqrt{N}\frac{h}{2}\right\}\right)\le\mu\left(\left\{|z|>\frac{1}{2}\right\}\right),$$

which is finite since μ satisfies (A_μ). Moreover, for $h>0$ small enough,

$$\begin{aligned}&\int_{|z|\le 1}|z|^2\,\mathrm{d}\nu_h(z)\\ &\le\sum_{0<|z_\alpha|\le 1}\int_{z_\alpha+R_h}|z_\alpha|^2\,\mathrm{d}\mu(z)\le\sum_{0<|z_\alpha|\le 1}\int_{z_\alpha+R_h}\left(|z|+\sqrt{N}\frac{h}{2}\right)^2\mathrm{d}\mu(z)\\ &\le\int_{h/2\le|z|\le 1+\sqrt{N}\frac{h}{2}}\left(|z|+\sqrt{N}\frac{h}{2}\right)^2\mathrm{d}\mu(z)\le\left(1+\sqrt{N}\right)^2\int_{|z|\le 2}|z|^2\,\mathrm{d}\mu(z),\end{aligned}$$

which is also finite since μ satisfies (A_μ). The proof is complete. □

Lemma 4.5.4. *Assume* (A_μ), $\mathcal{L}^\mu$ *and* $\mathcal{L}^\mu_h$ *are defined in* (4.1.3) *and* (4.2.10) *respectively, and* $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$*. Then*

$$\|\mathcal{L}^\mu_h[\psi] - \mathcal{L}^\mu[\psi]\|_{L^1} \to 0 \qquad as \qquad h \to 0^+.$$

Proof. The following inequality is just a use of the definitions,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\mathcal{L}^\mu_h[\psi](x) - \mathcal{L}^\mu[\psi](x)\right| \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left| \sum_{\alpha\neq 0} (\psi(x+z_\alpha)-\psi(x)) \int_{z_\alpha+R_h} \mathrm{d}\mu(z) - \sum_{\alpha\in\mathbb{Z}^N} \int_{z_\alpha+R_h} \psi(x+z)-\psi(x)\,\mathrm{d}\mu(z) \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \left(\left| \int_{R_h} \psi(x+z)-\psi(x)\,\mathrm{d}\mu(z)\right| + \left| \sum_{\alpha\neq 0} \int_{z_\alpha+R_h} \psi(x+z_\alpha)-\psi(x+z)\,\mathrm{d}\mu(z)\right| \right) \mathrm{d}x.
\end{aligned}$$

We will show that both terms go to zero with h. Indeed, for $|z| \leq 1$ we have that $|z|^2 \mathbf{1}_{R_h}(z) \to 0$ pointwise as $h \to 0^+$. Then, by Lebesgue's dominated convergence theorem, (A_μ), and Lemma 4.3.5 (b), we have as $h \to 0^+$

$$\int_{\mathbb{R}^N} \left| \int_{R_h} \psi(x+z)-\psi(x)\,\mathrm{d}\mu(z)\right| \mathrm{d}x \leq \frac{1}{2}\|D^2\psi\|_{L^1} \int_{|z|\leq 1} |z|^2 \mathbf{1}_{R_h}(z)\,\mathrm{d}\mu(z) \to 0.$$

For the second term, we need to consider separately the cases when when we are close or far from the origin. First note that for any $z \in z_\alpha + R_h$ we have that $|z_\alpha - z| \leq \sqrt{N}\frac{h}{2}$. Since μ satisfies (A_μ) and $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
I_{\mathrm{ext}} &:= \int_{\mathbb{R}^N} \left| \sum_{|\alpha|h>1} \int_{z_\alpha+R_h} \psi(x+z_\alpha)-\psi(x+z)\,\mathrm{d}\mu(z)\right| \mathrm{d}x \\
&\leq \|D\psi\|_{L^1(\mathbb{R}^N)} \sum_{|\alpha|h>1} \int_{z_\alpha+R_h} |z_\alpha - z|\,\mathrm{d}\mu(z) \\
&\leq h\frac{\sqrt{N}}{2} \|D\psi\|_{L^1(\mathbb{R}^N)} \int_{|z|>1/2} \mathrm{d}\mu(z) \to 0 \quad \text{as} \quad h \to 0^+.
\end{aligned}$$

On the other hand, by the symmetry of μ and also of the term in the sum, we have that

$$\begin{aligned}
I_{\mathrm{int}} &:= \int_{\mathbb{R}^N} \left| \sum_{0<|\alpha|h\leq 1} \int_{z_\alpha+R_h} \psi(x+z_\alpha)-\psi(x+z)\,\mathrm{d}\mu(z)\right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \left| \sum_{0<|\alpha|h\leq 1} \int_{z_\alpha+R_h} \psi(x+z_\alpha)-\psi(x+z)-(z_\alpha - z)\cdot D\psi(x)\,\mathrm{d}\mu(z)\right| \mathrm{d}x.
\end{aligned}$$

We make use of the Taylor expansions

$$\begin{aligned}
\psi(x+z_\alpha) &= \psi(x+z) + D\psi(x+z)\cdot(z_\alpha - z) + G_1(x,z,z_\alpha)(z_\alpha - z)\cdot(z_\alpha - z)\\
D\psi(x+z) &= D\psi(x) + G_2(x,z)\cdot z
\end{aligned}$$

where $\|G_1\|_{L^1(\mathbb{R}^N,\mathrm{d}x)} + \|G_2\|_{L^1(\mathbb{R}^N,\mathrm{d}x)} \leq C\|D^2\psi\|_{L^1(\mathbb{R}^N)}$ for some constant $C>0$. In this way,

$$\begin{aligned}
I_{\text{int}} &\leq \int_{\mathbb{R}^N}\left|\sum_{0<|\alpha|h\leq 1}\int_{z_\alpha+R_h} G_2 z\cdot(z_\alpha - z) + G_1(z_\alpha - z)\cdot(z_\alpha - z)\,\mathrm{d}\mu(z)\right|\mathrm{d}x\\
&= C\|D^2\psi\|_{L^1(\mathbb{R}^N)}\sum_{0<|\alpha|h\leq 1}\int_{z_\alpha+R_h}|z||z_\alpha - z| + |z_\alpha - z|^2\,\mathrm{d}\mu(z)\\
&\leq C\|D^2\psi\|_{L^1(\mathbb{R}^N)}\int_{\frac{h}{2}<|z|\leq 1+\sqrt{N}\frac{h}{2}}\frac{h}{2}\left(|z|+\frac{h}{2}\right)^2\mathrm{d}\mu(z)\to 0 \quad \text{as} \quad h\to 0^+.
\end{aligned}$$

Since the integrand is dominated by $2|z|^2$ which is an integrable function with respect to the measure μ on the set $\{z\in\mathbb{R}^N : |z|\leq 1\}$ by (A_μ), the last term goes to zero by Lebesgue's dominated convergence theorem. □

Proof of Proposition 4.2.15. Note that by Lemmas 4.5.2 and 4.5.3, L_h^σ and $\mathcal{L}_h^\mu$ are in the class of operators defined by (4.1.3) and (A_μ).

(a) Existence, uniqueness and regularity follow from Theorem 4.2.8.

(b) Follows from Theorem 4.2.9 (c) and (d) and interpolation.

(c) Lemmas 4.5.2 and 4.5.4 ensure the L^1-consistency.

(d) Follows from Theorem 4.2.9 (b).

(e) Follows from Theorem 4.2.9 (f). □

Proof of Proposition 4.2.16. By Lemmas 4.5.2 and 4.5.3 and Proposition 4.2.15 (c),

$$\sup_h \int_{|z|>0}\min\{|z|^2,1\}\,\mathrm{d}(\mu_{h,\sigma}+\nu_h)(z) < \infty,$$

and

$$\|\left(L_h^\sigma + \mathcal{L}_h^\mu\right)[\psi] - \left(L^\sigma + \mathcal{L}^\mu\right)[\psi]\|_{L^1} \to 0 \qquad \text{as} \qquad h\to 0^+.$$

Since also φ and u_0 are fixed (that is, independent of h), by Theorem 4.2.11 there is a subsequence $\{u_{h_n}\}_{n\in\mathbb{N}}$ of solutions of (4.2.5), that converge in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ to a function u. Moreover, this function u is a distributional solution of (4.2.6). Finally, u also belongs to $L^\infty(Q_T)\cap L^1(Q_T)$ by Proposition 4.2.15 (b) and Fatou's lemma. □

Proof of Corollary 4.2.17. Any limit point u from Proposition 4.2.16 is a distributional solution of (4.2.7) and (4.1.2). □

Proof of Theorem 4.2.18. By Proposition 4.2.16 there is a converging subsequence with a limit u which has the right regularity and is a distributional solution of (4.2.6). Assume there is a subsequence that converge to another limit v. Then by Proposition 4.2.16 again, there is a subsubsequence that converge to a limit which is a distributional solution. By uniqueness of the limit, v is a distributional solution. But then $v=u$ by the uniqueness given in Corollary 4.2.4 for the case $\sigma\equiv 0$ or the local result in [19]. Hence all subsequence limits are equal to u and since the sequence itself is bounded (Proposition 4.2.15 (b)), the whole sequence converges to u. □

4.6 Auxiliary elliptic equation

In this section we study the elliptic equation (4.3.1) introduced in Section 4.3 with the ultimate goal to prove Theorems 4.3.1 and 4.3.4. We will also need the following approximation of (4.3.1) where the measure μ is replaced by $\mu_r := \mathbf{1}_{|z|>r}\mu$:

$$\varepsilon v_{\varepsilon,r}(x) - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}](x) = g(x) \quad \text{in} \quad \mathbb{R}^N, \tag{4.6.1}$$

with $\varepsilon>0$,

$$\mathcal{L}^{\mu_r}[\psi](x) = \int_{|z|>0} \psi(x+z)-\psi(x)\,\mathrm{d}\mu_r(z).$$

Note that for any $r>0$, the operator $\mathcal{L}^{\mu_r}[\psi]$ is well-defined for merely bounded ψ, and that Lemma 4.3.5 also holds for $\mathcal{L}^{\mu_r}$; see Remark 4.3.6 (b). Also recall the notation $B^{\mu}_{\varepsilon} = (\varepsilon I - \mathcal{L}^{\mu})^{-1}$ and define $B^{\mu_r}_{\varepsilon} := (\varepsilon I - \mathcal{L}^{\mu_r})^{-1}$.

Remark 4.6.1. (a) Since (4.3.1) and (4.6.1) are linear equations, we have formally, for any multiindex $\alpha\in\mathbb{N}^N$, that $D^\alpha v$ is a solution of (4.3.1) or (4.6.1) with right hand side $D^\alpha g$ if v is a solution of the same equation with right hand side g.

(b) Let $\psi\in C^2_{\mathrm{b}}(\mathbb{R}^N)$, and let $p\in\{1,\infty\}$. Since

$$\left(\mathcal{L}^{\mu}-\mathcal{L}^{\mu_r}\right)[\psi](x) = \int_{|z|\leq r} \psi(x+z)-\psi(x)-z\cdot D\psi(x)\,\mathrm{d}\mu(z),$$

we have that $\mathcal{L}^{\mu_r}[\psi] \to \mathcal{L}^{\mu}[\psi]$ in $L^p(\mathbb{R}^N)$ as $r \to 0^+$ by Lemma 4.3.5 (b) and Lebesgue's dominated convergence theorem.

4.6.1 Preliminary results

We will state and prove a very general Stroock-Varopoulos type of inequality which is of independent interest. First we consider the bounded operators $\mathcal{L}^{\mu_r}$.

Lemma 4.6.2. *Assume* (A_μ), $\psi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ *and* $\zeta \in C(\mathbb{R})$ *is nondecreasing. Then for any* $r > 0$ *we have,*

$$\begin{aligned} I_r &:= \int_{\mathbb{R}^N} \zeta(\psi(x))\mathcal{L}^{\mu_r}[\psi](x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z|>0} \big(\zeta(\psi(x+z)) - \zeta(\psi(x))\big)\big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\mu_r(z)\,\mathrm{d}x, \end{aligned}$$

and in particular, $I_r \leq 0$.

Remark 4.6.3. More generally, the above lemma holds as long as the integral I_r is well-defined for ψ and $\zeta(\psi)$.

In the proof we need a technical lemma which wil be proved in Appendix 4.7.

Lemma 4.6.4. *Assume* ν *is a nonnegative, symmetric and locally finite Borel measure on* $\mathbb{R}^N$. *Let* A, B *be Borel sets on* $\mathbb{R}^N$, *and let*

$$M_1(A,B) = \int_A \left(\int_{B-z} \mathrm{d}\nu(x)\right)\,\mathrm{d}z = \int_A \nu(B-z)\,\mathrm{d}z.$$

$$M_2(A,B) = \int_B \left(\int_{A-z} \mathrm{d}\nu(x)\right)\,\mathrm{d}z = \int_B \nu(A-z)\,\mathrm{d}z.$$

Then $M_1(A,B) = M_2(A,B)$.

Proof of Lemma 4.6.2. Observe that $\zeta(\psi) \in L^\infty(\mathbb{R}^N)$, and since $\int_{\mathbb{R}^N} |\mathcal{L}^{\mu_r}[\psi]|\,\mathrm{d}x \leq 2\|\psi\|_{L^1}\int_{|z|>r}\mathrm{d}\mu(z)$, $\mathcal{L}^{\mu_r}[\psi] \in L^1(\mathbb{R}^N)$. Hence I_r is well-defined.

By the symmetry of μ, the gradient term in the nonlocal operator vanishes. Fubini's theorem and a relabelling of the variables gives

$$\begin{aligned} I_r &= \int_{\mathbb{R}^N} \zeta(\psi(x)) \int_{|z|>0} \psi(x+z) - \psi(x)\,\mathrm{d}\mu_r(z)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{|z-x|>0} \zeta(\psi(x))\big(\psi(z)) - \psi(x)\big)\mathbf{1}_{|z-x|>r}\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{|x-z|>0} \zeta(\psi(z))\big(\psi(x) - \psi(z)\big)\mathbf{1}_{|x-z|>r}\,\mathrm{d}\mu_{-z}(x)\,\mathrm{d}z. \end{aligned}$$

Since $\mathrm{d}\mu(z)$ is a nonnegative, symmetric and finite Radon measure on $\mathbb{R}^N$ (and hence a Borel measures), we can use Lemma 4.6.4 to see that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{|x-z|>0}\zeta(\psi(z))\big(\psi(x)-\psi(z)\big)\mathbf{1}_{|x-z|>r}\,\mathrm{d}\mu_{-z}(x)\,\mathrm{d}z\\&=\int_{\mathbb{R}^N}\int_{|x-z|>0}\zeta(\psi(z))\big(\psi(x)-\psi(z)\big)\mathbf{1}_{|x-z|>r}\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x.\end{aligned}$$

It then follows that

$$\begin{aligned}2I_r&=-\int_{\mathbb{R}^N}\int_{|z-x|>0}\zeta(\psi(x))\big(\psi(x))-\psi(z)\big)\mathbf{1}_{|z-x|>r}\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}\int_{|x-z|>0}\zeta(\psi(z))\big(\psi(x)-\psi(z)\big)\mathbf{1}_{|x-z|>r}\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x\\&=-\int_{\mathbb{R}^N}\int_{|z-x|>0}\big(\zeta(\psi(x))-\zeta(\psi(z))\big)\big(\psi(x)-\psi(z)\big)\mathbf{1}_{|z-x|>r}\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x.\end{aligned}$$

Since $\big(\zeta(\psi(x))-\zeta(\psi(z))\big)\big(\psi(x)-\psi(z)\big)\geq 0$ for all $x,z\in\mathbb{R}^N$, $I_r\leq 0$. □

Now we give the general result, considering the general nonlocal operator $\mathcal{L}^\mu$.

Corollary 4.6.5 (General Stroock-Varopoulos). *Assume* (A_μ), *and* $\zeta\in C^1(\mathbb{R})$ *such that* $\zeta'\geq 0$.

(a) Let $\psi\in C_{\mathrm{b}}^1(\mathbb{R}^N)\cap W^{2,1}(\mathbb{R}^N)$. *Then*

$$\begin{aligned}I&:=\int_{\mathbb{R}^N}\zeta(\psi(x))\mathcal{L}^\mu[\psi](x)\,\mathrm{d}x\\&=-\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0}\big(\zeta(\psi(z))-\zeta(\psi(x))\big)\big(\psi(z)-\psi(x)\big)\,\mathrm{d}\mu(z)\,\mathrm{d}x\\&\leq 0.\end{aligned}$$

(b) Let $\psi\in C_{\mathrm{b}}^2(\mathbb{R}^N)\cap W^{2,1}(\mathbb{R}^N)$. *If* $Z\in C^2(\mathbb{R})$ *is such that* $Z(0)=0$ *and* $(Z')^2=\zeta'$, *then*

$$\int_{\mathbb{R}^N}\zeta(\psi(x))\mathcal{L}^\mu[\psi](x)\,\mathrm{d}x\leq-\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0}\big(Z(\psi(z))-Z(\psi(x))\big)^2\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x.$$

Moreover,

$$\begin{aligned}\Big(Z(\psi),\mathcal{L}^\mu[Z(\psi)]\Big)_{L^2(\mathbb{R}^N)}&=-\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0}\big(Z(\psi(z))-Z(\psi(x))\big)^2\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x\\&=-\Big\|(\mathcal{L}^\mu)^{\frac{1}{2}}[Z(\psi)]\Big\|_{L^2}^2.\end{aligned}$$

Remark 4.6.6. The (energy) norm in part (b) is much studied when $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, $s \in (0,2)$, and $Z = I$ (see [7, 42]). In this case

$$\Big(\psi, (-\Delta)^{\frac{s}{2}}\psi\Big)_{L^2(\mathbb{R}^N)} = \frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0} \frac{\big(\psi(z)-\psi(x)\big)^2}{|z-x|^{N+s}}\,\mathrm{d}z\,\mathrm{d}x = \Big\|(-\Delta)^{\frac{s}{2}}\psi\Big\|^2_{L^2(\mathbb{R}^N)}.$$

This is called the Gagliardo (semi)norm of ψ and is denoted by $[\psi]_{W^{\frac{s}{2},2}(\mathbb{R}^N)}$.

Proof. (a) By Remark 4.6.1 (b),

$$\left|\int_{\mathbb{R}^N} \zeta(\psi)(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[\psi]\,\mathrm{d}x\right| \leq \|\zeta(\psi)\|_{L^\infty}\|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[\psi]\|_{L^1} \to 0 \qquad \text{as} \qquad r \to 0^+,$$

and we may send $r \to 0^+$ in Lemma 4.6.2 to get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \zeta(\psi(x))\mathcal{L}^\mu[\psi](x)\,\mathrm{d}x \\
&= -\frac{1}{2}\lim_{r\to 0^+}\int_{\mathbb{R}^N}\int_{|z-x|>r} \big(\zeta(\psi(z)) - \zeta(\psi(x))\big)\big(\psi(z)-\psi(x)\big)\,\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x.
\end{aligned}$$

By the assumptions on ζ, ψ and (A_μ), $\big(\zeta(\psi(z))-\zeta(\psi(x))\big)\big(\psi(z)-\psi(x)\big) \geq 0$ is integrable with respect to $\mathrm{d}\mu_{-x}(z)\,\mathrm{d}x$ on $\mathbb{R}^N \times \mathbb{R}^N\backslash\{0\}$ since

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|\int_{|z-x|>0} \big(\zeta(\psi(z)) - \zeta(\psi(x))\big)\big(\psi(z)-\psi(x)\big)\,\mathrm{d}\mu_{-x}(z)\right|\mathrm{d}x \\
&\leq \|\zeta'(\psi)\|_{L^\infty}\|D\psi\|_{L^\infty}\|D\psi\|_{L^1}\int_{|z|\leq 1}|z|^2\,\mathrm{d}\mu(z) \\
&\qquad + 4\|\zeta(\psi)\|_{L^\infty}\|\psi\|_{L^1}\int_{|z|>1}\mathrm{d}\mu(z).
\end{aligned} \tag{4.6.2}$$

Thus, Lebesgue's dominated convergence theorem gives the desired result.

(b) For $a, b \in \mathbb{R}$, the Fundamental Theorem of Calculus and Jensen's inequality gives the following pointwise inequality:

$$\begin{aligned}
(Z(b) - Z(a))^2 &= \left(\int_a^b Z'(t)\,\mathrm{d}t\right)^2 \leq (b-a)\int_a^b \big(Z'(t)\big)^2\,\mathrm{d}t \\
&= (b-a)\int_a^b \zeta'(t)\,\mathrm{d}t = (b-a)(\zeta(b)-\zeta(a)).
\end{aligned} \tag{4.6.3}$$

By the assumptions, we can easily check that $Z(\psi) \in C_\mathrm{b}^2(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$. So the integral

$$-\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0} \big(Z(\psi(z)) - Z(\psi(x))\big)^2 d\mu_{-x}(z)\,\mathrm{d}x$$

is well-defined using a similar argument as in (4.6.2). Then, part (a) and (4.6.3) gives the first result of part (b).

Next, part (a) yields

$$\Big(Z(\psi),\mathcal{L}^{\mu}[Z(\psi)]\Big)_{L^2(\mathbb{R}^N)}=-\frac{1}{2}\int_{\mathbb{R}^N}\int_{|z-x|>0}\big(Z(\psi(z))-Z(\psi(x))\big)^2 d\mu_{-x}(z)\,\mathrm{d}x.$$

Moreover, since $Z(\psi)\in C_{\mathrm{b}}^2(\mathbb{R}^N)\cap W^{2,1}(\mathbb{R}^N)$, then by Lemma 4.3.5 (b) and interpolation, both $Z(\psi)$ and $\mathcal{L}^{\mu}[Z(\psi)]$ are in $L^2(\mathbb{R}^N)$. We then conclude the proof by application of Lemma 4.3.7 and Remark 4.3.8 (b). □

4.6.2 Results for the approximate elliptic equation (4.6.1)

We will now focus on proving some a priori, uniqueness, existence, and stability results for (4.6.1).

Proposition 4.6.7. *Assume* (A_μ).

(a) If $g\in L^\infty(\mathbb{R}^N)$ and $v_{\varepsilon,r}\in L^\infty(\mathbb{R}^N)$ solves $\varepsilon v_{\varepsilon,r}-\mathcal{L}^{\mu_r}[v_{\varepsilon,r}]\leq g$ a.e., then

$$\varepsilon\|(v_{\varepsilon,r})^+\|_{L^\infty(\mathbb{R}^N)}\leq\|(g)^+\|_{L^\infty(\mathbb{R}^N)}.$$

(b) If $g\in L^\infty(\mathbb{R}^N)$ and $v_{\varepsilon,r}\in L^\infty(\mathbb{R}^N)$ is an a.e. solution of (4.6.1), then

$$\varepsilon\|v_{\varepsilon,r}\|_{L^\infty(\mathbb{R}^N)}\leq\|g\|_{L^\infty(\mathbb{R}^N)}.$$

(c) Let $g,\hat{g},v_{\varepsilon,r},\hat{v}_{\varepsilon,r}\in L^\infty(\mathbb{R}^N)$, $\varepsilon v_{\varepsilon,r}-\mathcal{L}^{\mu_r}[v_{\varepsilon,r}]\leq g$ a.e. and $\varepsilon\hat{v}_{\varepsilon,r}-\mathcal{L}^{\mu_r}[\hat{v}_{\varepsilon,r}]\geq\hat{g}$ a.e. If $g\leq\hat{g}$ a.e., then $v_{\varepsilon,r}\leq\hat{v}_{\varepsilon,r}$ a.e.

Proof. (a) Assume first that $g,v_{\varepsilon,r}\in C_{\mathrm{b}}(\mathbb{R}^N)$. Then for all $\delta>0$ there exists a $x_\delta\in\mathbb{R}^N$ such that

$$v_{\varepsilon,r}(x_\delta)+\delta>\sup\{v_{\varepsilon,r}\}.$$

Then, since $v_{\varepsilon,r}$ is an a.e. solution,

$$\begin{aligned}\varepsilon v_{\varepsilon,r}(x_\delta)&\leq g(x_\delta)+\int_{|z|>0}v_{\varepsilon,r}(x_\delta+z)-v_{\varepsilon,r}(x_\delta)\,\mathrm{d}\mu_r(z)\\&\leq\|(g)^+\|_{L^\infty(\mathbb{R}^N)}+\int_{|z|>r}\sup\{v_{\varepsilon,r}\}-v_{\varepsilon,r}(x_\delta)\,\mathrm{d}\mu(z)\\&\leq\|(g)^+\|_{L^\infty(\mathbb{R}^N)}+\delta\mu(\{z\in\mathbb{R}^N:|z|>r\}).\end{aligned}$$

Hence,

$$\varepsilon \sup\{v_{\varepsilon,r}\} < \varepsilon v_{\varepsilon,r}(x) + \varepsilon\delta \leq \|(g)^+\|_{L^\infty(\mathbb{R}^N)} + \delta\big(\varepsilon + \mu(\{z \in \mathbb{R}^N : |z| > r\})\big),$$

and we pass to the limit as $\delta \to 0^+$ to get

$$\varepsilon \sup\{v_{\varepsilon,r}\} \leq \|(g)^+\|_{L^\infty}.$$

In the general case, when $g, v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$, we need a regularization argument. Let $v_{\varepsilon,r}^\delta := \omega_\delta * v_{\varepsilon,r}$ and mollify the inequality to see that

$$\varepsilon v_{\varepsilon,r}^\delta - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}^\delta] \leq g_\delta \qquad \text{in} \qquad \mathbb{R}^N.$$

By the first part of the proof and the properties of mollifiers,

$$v_{\varepsilon,r}(x) \leq |v_{\varepsilon,r}^\delta(x) - v_{\varepsilon,r}(x)| + v_{\varepsilon,r}^\delta(x) \leq o(1) + \frac{1}{\varepsilon}\|(g)^+\|_{L^\infty(\mathbb{R}^N)} \quad \text{as} \quad \delta \to 0^+$$

for a.e. $x \in \mathbb{R}^N$. Part (a) follows.

(b) In a similar way as in (a), we find that $\varepsilon \sup\{-v_{\varepsilon,r}\} \leq \|(g)^-\|_{L^\infty(\mathbb{R}^N)}$ and combine with (a) to conclude that $\varepsilon\|v_{\varepsilon,r}\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}$.

(c) Since $w = v_{\varepsilon,r} - \hat{v}_{\varepsilon,r}$ solves $\varepsilon w - \mathcal{L}^{\mu_r}[w] \leq g - \hat{g}$, by (a) and the assumptions, it follows that $\varepsilon \sup\{w\} \leq \|(g-\hat{g})^+\|_{L^\infty(\mathbb{R}^N)} = 0$. □

Proposition 4.6.8 (Existence and uniqueness). *Assume* (A_μ).

(a) If $g \in C_\mathrm{b}(\mathbb{R}^N)$, then there exists a unique classical solution $v_{\varepsilon,r} \in C_\mathrm{b}(\mathbb{R}^N)$ of (4.6.1).

(b) If $g \in L^\infty(\mathbb{R}^N)$, then there exists a unique a.e. solution $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ of (4.6.1).

(c) If $g \in L^1(\mathbb{R}^N)$, then there exists a unique a.e. solution $v_{\varepsilon,r} \in L^1(\mathbb{R}^N)$ of (4.6.1).

(d) If $g \in C_\mathrm{b}^\infty(\mathbb{R}^N)$, then there exists a unique classical solution $v_{\varepsilon,r} \in C_\mathrm{b}^\infty(\mathbb{R}^N)$ of (4.6.1). *Moreover,*

$$\varepsilon\|D^\alpha v_{\varepsilon,r}\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty}$$

for each multiindex $\alpha \in \mathbb{N}^N$.

Proof. Note that $\mathcal{L}^{\mu_r}$ is a bounded linear operator on any one of the spaces $C_\mathrm{b}(\mathbb{R}^N)$, $L^\infty(\mathbb{R}^N)$, and $L^1(\mathbb{R}^N)$. The proofs of (a), (b), and (c) then follow from a standard fixed point argument using Banach's fixed point theorem; the full details can be found in [4].

(d) Let $v_{\varepsilon,r} = B_\varepsilon^{\mu_r}[g]$ and define $\delta_{i,h}\psi$ by

$$\delta_{i,h}\psi(x) = \frac{\psi(x+he_i)-\psi(x)}{h}.$$

By part (a), we have uniqueness for $C_\mathrm{b}(\mathbb{R}^N)$ solutions of (4.6.1). Hence, $v_{\varepsilon,r}(x+he_i) = B_\varepsilon^{\mu_r}[g(\cdot+he_i)](x)$, and then by uniqueness and linearity $\delta_{i,h}v = B_\varepsilon^{\mu_r}[\delta_{i,h}g]$. In addition, there exists a unique $w_{i,\varepsilon,r} \in C_\mathrm{b}(\mathbb{R}^N)$ such that $w_{i,\varepsilon,r} = B_\varepsilon^{\mu_r}[\partial_{x_i}g]$.

Using linearity and Proposition 4.6.7 (b), we get

$$\varepsilon\|w_{i,\varepsilon,r} - \delta_{i,h}v_{\varepsilon,r}\|_{L^\infty} \leq \|\partial_{x_i}g - \delta_{i,h}g\|_{L^\infty}.$$

When $h \to 0^+$, $\delta_{i,h}g \to \partial_{x_i}g$ uniformly on $\mathbb{R}^N$, and hence $\delta_{i,h}v_{\varepsilon,r} \to w_{i,\varepsilon,r}$ in L^∞. This implies that $\partial_{x_i}v_{\varepsilon,r} = w_{i,\varepsilon,r}$. Moreover, by Proposition 4.6.7 (b),

$$\|\partial_{x_i}v_{\varepsilon,r}\|_{L^\infty} = \|w_{i,\varepsilon,r}\|_{L^\infty} \leq \|\partial_{x_i}g\|_{L^\infty}.$$

A similar argument shows that for each multiindex $\alpha \in \mathbb{N}^N$, $D^\alpha v_{\varepsilon,r} = B_\varepsilon^{\mu_r}[D^\alpha g]$, and hence belongs to $C_\mathrm{b}(\mathbb{R}^N)$. □

Corollary 4.6.9. *Assume* (A_μ) *and* $g \in C_\mathrm{b}(\mathbb{R}^N)$. *If* $(g)^+ \in L^1(\mathbb{R}^N)$, *then* $(B_\varepsilon^{\mu_r}[g])^+ \in L^1(\mathbb{R}^N)$.

Proof. Note that $g \in C_\mathrm{b}(\mathbb{R}^N)$ implies that $(g)^+ \in C_\mathrm{b}(\mathbb{R}^N)$. By Proposition 4.6.8 (a) and (c), and the assumption on $(g)^+$, we have that have that $B_\varepsilon^{\mu_r}[g] \in C_\mathrm{b}(\mathbb{R}^N)$ and $B_\varepsilon^{\mu_r}[(g)^+] \in L^1(\mathbb{R}^N) \cap C_\mathrm{b}(\mathbb{R}^N)$ are the unique classical solutions of (4.6.1) with right-hand sides $g, (g)^+$, respectively. Proposition 4.6.7 (c) ensures that $B_\varepsilon^{\mu_r}[(g)^+] \geq 0$ since $(g)^+ \geq 0$. In the same way, we get $B_\varepsilon^{\mu_r}[-(g)^-] \in C_\mathrm{b}(\mathbb{R}^N)$ and $B_\varepsilon^{\mu_r}[-(g)^-] \leq 0$.

Adding the equations for $B_\varepsilon^{\mu_r}[(g)^+]$ and $B_\varepsilon^{\mu_r}[-(g)^-]$, and noting that $(g)^+ - (g)^- = g \in C_\mathrm{b}(\mathbb{R}^N)$, we get

$$\varepsilon\Big(B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-]\Big) - \mathcal{L}^{\mu_r}\Big[B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-]\Big] = g.$$

It follows that $B_\varepsilon^{\mu_r}[g] = B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-]$ by uniqueness. We conclude that $0 \leq (B_\varepsilon^{\mu_r}[g])^+ \leq B_\varepsilon^{\mu_r}[(g)^+]$, and thus, $(B_\varepsilon^{\mu_r}[g])^+ \in L^1(\mathbb{R}^N)$. □

4.6.3 Results for the elliptic equation (4.3.1)

Now, we state and prove comparison, uniqueness and existence results for classical solutions of (4.3.1). These results will be obtained from the corresponding results for (4.6.1) and limit procedures.

Lemma 4.6.10 (Comparison)**.** *Assume* (A_μ)*,* $g, \hat{g} \in L^\infty(\mathbb{R}^N)$*, and* $v_\varepsilon, \hat{v}_\varepsilon \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *are solutions of* (4.3.1) *with right-hand sides* $g, \hat{g}$ *respectively. If* $g \leq \hat{g}$ *a.e., then* $v_\varepsilon \leq \hat{v}_\varepsilon$ *in* $\mathbb{R}^N$*.*

Proof. Note that $w = v_\varepsilon - \hat{v}_\varepsilon$ solves $\varepsilon w - \mathcal{L}^\mu[w] \leq 0$, and hence, also

$$\varepsilon w - \mathcal{L}^{\mu_r}[w] \leq \|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)}.$$

By Proposition 4.6.7 (a), it then follows that

$$\varepsilon\|(w)^+\|_{L^\infty(\mathbb{R}^N)} \leq \|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)}.$$

Assume for moment that $w \in C_{\mathrm{b}}^2(\mathbb{R}^N)$. Then by remark 4.6.1 (b),

$$\|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)} \to 0 \qquad \text{as} \qquad r \to 0^+,$$

and we conclude that $w \leq 0$.

The general case follows by mollification: $w_\delta = w * \omega_\delta$ (cf. (4.1.7)) satisfies $\varepsilon w_\delta - \mathcal{L}^\mu[w_\delta] \leq 0$ and hence by the first part of the proof and properties of mollifiers,

$$w(x) \leq w_\delta(x) + |w(x) - w_\delta(x)| \leq 0 + o(1) \qquad \text{as} \qquad \delta \to 0^+$$

for every $x \in \mathbb{R}^N$. The proof is complete. □

Corollary 4.6.11 (Uniqueness)**.** *Assume* (A_μ)*, and* $g \in L^\infty(\mathbb{R}^N)$*. Then there is at most one classical solution* $v_\varepsilon \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of* (4.3.1)*.*

Proof. If $g = \hat{g}$ a.e., then Lemma 4.6.10 gives $v_\varepsilon = \hat{v}_\varepsilon$ in $\mathbb{R}^N$. □

Proposition 4.6.12 (Existence and Stability)**.** *Assume* (A_μ)*,* $g \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$*,* $\varepsilon > 0$*.*

(a) There exists a unique classical solution $B_\varepsilon^\mu[g] = v_\varepsilon \in C_{\mathrm{b}}^2$ *of* (4.3.1)*.*

(b) Any sequence $\{v_{\varepsilon,r_n}\}_{n\in\mathbb{N}}$ *of solutions of* (4.6.1) *converges locally uniformly to* $v_\varepsilon = B_\varepsilon^\mu[g]$ *of part (a) as* $r_n \to 0^+$*.*

Proof. (a) Let $0 < r_n \to 0^+$ as $n \to \infty$, and let $v_n := v_{\varepsilon,r_n} \in C_\mathrm{b}^\infty(\mathbb{R}^N)$ be the unique solution of (4.6.1) given by Proposition 4.6.8 (d). Moreover, for all $n > 0$,

$$
\begin{aligned}
\varepsilon\|v_n\|_{L^\infty} &\leq \|g\|_{L^\infty}, &\qquad \varepsilon\|Dv_n\|_{L^\infty} &\leq \|Dg\|_{L^\infty},\\
\varepsilon\|D^2v_n\|_{L^\infty} &\leq \|D^2g\|_{L^\infty}, &\qquad \varepsilon\|D^3v_n\|_{L^\infty} &\leq \|D^3g\|_{L^\infty}.
\end{aligned}
$$

The sequences $\{v_n\}_{n>0}$, $\{Dv_n\}_{n>0}$ and $\{D^2v_n\}_{n>0}$ are thus equibounded and equilipschitz. By Arzelà-Ascoli's theorem there exists a subsequence (still denoted by v_n, Dv_n and D^2v_n) such that (v_n, Dv_n, D^2v_n) converges locally uniformly (and hence a.e.) as $n \to \infty$ to a limit $(\overline{v}, \overline{Dv}, \overline{D^2v})$ which is bounded and continuous.

We check that $D\overline{v} = \overline{Dv}$ and $D^2\overline{v} = \overline{D^2v}$. Let $\alpha \in \mathbb{N}^N$ denote a multiindex. By Taylor's theorem

$$
\begin{aligned}
v_n(y) = &\,v_n(x) + Dv_n(x)\cdot(y-x) + \frac{1}{2}D^2v_n(x)(y-x)\cdot(y-x)\\
&+ \sum_{|\alpha|=3}\frac{3}{\alpha!}(y-x)^\alpha\int_0^1(1-t)^2D^\alpha v_n(x+t(y-x))\,\mathrm{d}t.
\end{aligned}
\tag{4.6.4}
$$

Since

$$
\left|\sum_{|\alpha|=3}\frac{3}{\alpha!}(y-x)^\alpha\int_0^1(1-t)^2D^\alpha v_n(x+t(y-x))\,\mathrm{d}t\right| \leq \frac{1}{\varepsilon}\|D^3g\|_{L^\infty}\sum_{|\alpha|=3}\frac{|y-x|^\alpha}{\alpha!},
$$

we can take the locally uniform limit in (4.6.4) as $n \to \infty$ to obtain that

$$
\overline{v}(y) = \overline{v}(x) + \overline{Dv}(x)\cdot(y-x) + \frac{1}{2}\overline{D^2v}(x)(y-x)\cdot(y-x) + o(|y-x|^2) \quad \text{as} \quad y \to x.
$$

By definition, it then follows that $D\overline{v} = \overline{Dv}$ and $D^2\overline{v} = \overline{D^2v}$.

We now go to the limit in (4.6.1) as $r_n \to 0^+$, and we may assume that $r_n < 1$. In order to show the convergence, the nonlocal operator in (4.6.1) will be written as

$$
\mathcal{L}^{\mu_{r_n}}[v_n](x) = \mathcal{L}_1^{\mu_{r_n}}[v_n](x) + \int_{|z|>1} v_n(x+z) - v_n(x)\,\mathrm{d}\mu(z),
$$

with

$$
\mathcal{L}_1^{\mu_{r_n}}[v_n](x) := \int_{|z|\leq 1} v_n(x+z) - v_n(x) - z\cdot Dv_n(x)\,\mathrm{d}\mu_{r_n}(z).
$$

By the triangle inequality and Lemma 4.3.5 (a),

$$\begin{aligned}
&\left|\mathcal{L}_1^{\mu_{r_n}}[v_n](x) - \mathcal{L}_1^{\mu}[\overline{v}](x)\right| \\
&\leq \left|\mathcal{L}_1^{\mu_{r_n}}[v_n - \overline{v}](x)\right| + \left|(\mathcal{L}_1^{\mu_{r_n}} - \mathcal{L}_1^{\mu})[\overline{v}](x)\right| \\
&\leq \frac{1}{2}\max_{|z|\leq 1}\left|D^2 v_n(x+z) - D^2\overline{v}(x+z)\right| \int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) \\
&\quad + \frac{1}{2}\max_{|z|\leq 1}\left|D^2\overline{v}(x+z)\right| \int_{|z|\leq 1} |z|^2 \mathbf{1}_{|z|\leq r_n} \,\mathrm{d}\mu(z).
\end{aligned}$$

So, the local uniform convergence and Lebesgue's dominated convergence theorem ensures that $\left|\mathcal{L}_1^{\mu_{r_n}}[v_n](x) - \mathcal{L}_1^{\mu}[\overline{v}](x)\right| \to 0$ as $r_n \to 0^+$ for all $x \in \mathbb{R}^N$. The remaining term in the nonlocal operator also converges by Lebesgue's dominated convergence theorem:

$$\int_{|z|>1} v_n(x+z) - v_n(x)\,\mathrm{d}\mu(z) \to \int_{|z|>1} \overline{v}(x+z) - \overline{v}(x)\,\mathrm{d}\mu(z) \qquad \text{as} \qquad r_n \to 0^+.$$

Sending $r_n \to 0^+$ in (4.6.1) then shows that $\overline{v}$ solves (4.3.1). Moreover, the limit is unique by Corollary 4.6.11.

(b) In fact, part (a) shows that all limit points of the sequences $\{v_{\varepsilon,r_n}\}_{n\in\mathbb{N}}$ coincide by uniqueness (see Corollary 4.6.11). By Proposition 4.6.8 (d), every sequence is bounded, and hence, the whole sequence converge locally uniformly to the solution of (4.3.1) as $r_n \to 0^+$. □

Proposition 4.6.13. *Assume* (A_μ), $g \in C_\mathrm{b}^\infty(\mathbb{R}^N)$, $\varepsilon > 0$, *and* $v_\varepsilon = B_\varepsilon^\mu[g]$. *If* $(g)^+ \in L^1(\mathbb{R}^N)$, *then*

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g)^+ \,\mathrm{d}x.$$

Proof. By Proposition 4.6.8 (d), for any $r > 0$, there exists a unique function $v_{\varepsilon,r} \in C_\mathrm{b}^\infty(\mathbb{R}^N)$ such that

$$\varepsilon v_{\varepsilon,r}(x) - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}](x) = g(x) \qquad \text{in} \qquad \mathbb{R}^N.$$

Consider $\mathcal{X} \in C_\mathrm{c}^\infty(\mathbb{R}^N)$ such that $0 \leq \mathcal{X}$,

$$\mathcal{X}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 2 \end{cases},$$

and define $\mathcal{X}_R(x) = \mathcal{X}(\frac{x}{R}) \in C_\mathrm{c}^\infty(\mathbb{R}^N)$ for $R > 0$. Then for every $r > 0$, by Proposition 4.6.8 (d), there exists a function $u_R \in C_\mathrm{b}^\infty(\mathbb{R}^N)$ such that

$$\varepsilon u_R(x) - \mathcal{L}^{\mu_r}[u_R](x) = g(x)\mathcal{X}_R(x) \qquad \text{for all} \qquad x \in \mathbb{R}^N. \tag{4.6.5}$$

Let $\zeta_\delta : \mathbb{R} \to \mathbb{R}_+$ be a smooth approximation of the sign$^+$ function. More precisely, $\zeta_\delta(x) = 0$ for $x \leq 0$, $\zeta_\delta'(x) \geq 0$ and $0 < \zeta_\delta(x) \leq 1$ for $x > 0$. Since $0 \leq (g\mathcal{X}_R)^+ \leq (g)^+ \in L^1(\mathbb{R}^N)$, $(u_R)^+ \in L^1(\mathbb{R}^N)$ by Corollary 4.6.9, and

$$\left|\int_{\mathbb{R}^N} u_R \zeta_\delta(u_R)\,\mathrm{d}x\right| \leq \|(u_R)^+\|_{L^1}\|\zeta_\delta(u_R)\|_{L^\infty}$$
$$\left|\int_{\mathbb{R}^N} g\mathcal{X}_R \zeta_\delta(u_R)\,\mathrm{d}x\right| \leq \|g\|_{L^\infty}\int_{|x|\leq 2R}\mathrm{d}x.$$

Then by (4.6.5), $\left|\int_{\mathbb{R}^N} \mathcal{L}^{\mu_r}[u_R]\zeta_\delta(u_R)\,\mathrm{d}x\right| < \infty$, and we may multiply (4.6.5) by ζ_δ and integrate over $\mathbb{R}^N$ to find that

$$\varepsilon\int_{\mathbb{R}^N} u_R\zeta_\delta(u_R)\,\mathrm{d}x = \underbrace{\int_{\mathbb{R}^N} \mathcal{L}^{\mu_r}[u_R]\zeta_\delta(u_R)\,\mathrm{d}x}_{I_r} + \int_{\mathbb{R}^N} g\mathcal{X}_R\zeta_\delta(u_R)\,\mathrm{d}x.$$

So, Lemma 4.6.2 and Remark 4.6.3 gives that $I_r \leq 0$ and hence

$$\varepsilon\int_{\mathbb{R}^N} u_R\zeta_\delta(u_R)\,\mathrm{d}x \leq \int_{\mathbb{R}^N} g\mathcal{X}_R\zeta_\delta(u_R)\,\mathrm{d}x \leq \int_{\mathbb{R}^N}(g)^+\,\mathrm{d}x.$$

Letting $\zeta_\delta(u_R) \to \mathrm{sign}^+(u_R)$ as $\delta \to 0^+$ in the above inequality (using Fatou's lemma on the left-hand side since $u_R\zeta_\delta(u_R) \geq 0$) yields

$$\varepsilon\int_{\mathbb{R}^N}(u_R)^+\,\mathrm{d}x \leq \int_{\mathbb{R}^N}(g)^+\,\mathrm{d}x. \tag{4.6.6}$$

We note that the sequence $\{u_R\}_{R>0}$ is equibounded and equilipschitz since Proposition 4.6.8 (d) gives

$$\|u_R\|_{L^\infty} \leq \frac{1}{\varepsilon}\|g\|_{L^\infty}$$
$$\|Du_R\|_{L^\infty} \leq \frac{1}{\varepsilon}\|D\left(g\mathcal{X}_R\right)\|_{L^\infty} \leq \frac{1}{\varepsilon}\|Dg\|_{L^\infty} + \frac{1}{\varepsilon}\mathcal{O}\left(\frac{1}{R}\right).$$

Hence, by Arcelá-Ascoli, $u_R \to u$ as $R \to \infty$ locally uniformly in $\mathbb{R}^N$ (and thus a.e. in $\mathbb{R}^N$). Sending $R \to \infty$ in (4.6.5) shows that $u = v_{\varepsilon,r}$; the unique solution of (4.3.1) given by Proposition 4.6.8 (d). Furthermore, we can send $R \to \infty$ in (4.6.6) (again using Fatou's lemma) to obtain

$$\varepsilon\int_{\mathbb{R}^N}(v_{\varepsilon,r})^+\,\mathrm{d}x \leq \int_{\mathbb{R}^N}(g)^+\,\mathrm{d}x.$$

By Fatou's lemma and Proposition 4.6.12 (b), we can let $r_n \to 0^+$ in the above estimate to get

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g)^+ \,\mathrm{d}x,$$

where v_ε is the classical solution of (4.3.1). □

Corollary 4.6.14. *Assume* (A_μ)*,* $g \in C_\mathrm{b}^\infty(\mathbb{R}^N)$*,* $\varepsilon > 0$*, and* $v_\varepsilon = B_\varepsilon^\mu[g]$*.*

(a) If $(g)^- \in L^1(\mathbb{R}^N)$*, then* $\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^- \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g)^- \,\mathrm{d}x$*.*

(b) If $g \in L^1(\mathbb{R}^N)$*, then* $\varepsilon \int_{\mathbb{R}^N} |v_\varepsilon| \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |g| \,\mathrm{d}x$

Proof. (a) Note that $(g)^- \in L^1(\mathbb{R}^N)$ implies that $(-g)^+ \in L^1(\mathbb{R}^N)$. Since $B_\varepsilon^\mu[-g] = -B_\varepsilon^\mu[g]$, we have by Proposition 4.6.13 that

$$\varepsilon \int_{R^N} (B_\varepsilon^\mu[g])^- \,\mathrm{d}x = \varepsilon \int_{\mathbb{R}^N} (B_\varepsilon^\mu[-g])^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (-g)^+ = \int_{\mathbb{R}^N} (g)^- \,\mathrm{d}x.$$

(b) Follows by noting that $(v_\varepsilon)^+ + (v_\varepsilon)^- = |v_\varepsilon|$. □

Below, we collect the main results for (4.3.1).

Theorem 4.6.15. *Assume* (A_μ)*,* $g \in L^1_\mathrm{loc}(\mathbb{R}^N)$*, and* $v_\varepsilon \in L^1_\mathrm{loc}(\mathbb{R}^N)$ *is a distributional solution of* (4.3.1)*.*

(a) If $(g)^+ \in L^1(\mathbb{R}^N)$*, then*

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g)^+ \,\mathrm{d}x.$$

(b) If $g \geq 0$ *a.e. on* $\mathbb{R}^N$*, then* $v_\varepsilon \geq 0$ *a.e. on* $\mathbb{R}^N$*.*

Proof. (a) Let $\omega_\delta \in C_\mathrm{c}^\infty(\mathbb{R}^N)$ be defined in (4.1.7), and let $v_{\varepsilon,\delta} = v_\varepsilon * \omega_\delta \in C_\mathrm{b}^\infty(\mathbb{R}^N)$. By assumption,

$$\varepsilon \int_{\mathbb{R}^N} v_\varepsilon \psi \,\mathrm{d}y - \int_{\mathbb{R}^N} v_\varepsilon \mathcal{L}^\mu[\psi] \,\mathrm{d}y = \int_{\mathbb{R}^N} g\psi \,\mathrm{d}y$$

for all $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. Taking $\psi(y) = \omega_\delta(x-y)$ for $x \in \mathbb{R}^N$, we get the pointwise equation

$$\varepsilon v_{\varepsilon,\delta} - \mathcal{L}^\mu[v_{\varepsilon,\delta}] = g * \omega_\delta \qquad \text{in} \qquad \mathbb{R}^N.$$

Note that $0 \leq (g * \omega_\delta)^+ \leq (g)^+ * \omega_\delta \in L^1(\mathbb{R}^N)$ (see e.g. Lemma 5.1 in [45]), so Proposition 4.6.13 gives

$$\varepsilon \int_{\mathbb{R}^N} (v_{\varepsilon,\delta})^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g * \omega_\delta)^+ \,\mathrm{d}x.$$

Then by Fatou's lemma

$$\varepsilon \int_{\mathbb{R}^N} \liminf_{\delta\to 0^+} (v_{\varepsilon,\delta})^+ \,\mathrm{d}x \leq \liminf_{\delta\to 0^+} \varepsilon \int_{\mathbb{R}^N} (v_{\varepsilon,\delta})^+ \,\mathrm{d}x \leq \liminf_{\delta\to 0^+} \int_{\mathbb{R}^N} (g)^+ * \omega_\delta \,\mathrm{d}x.$$

Since $(\cdot)^+$ is continuous, $(g)^+ \in L^1(\mathbb{R}^N)$, and $v_{\varepsilon,\delta} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, the properties of mollifiers yields

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (g)^+ \,\mathrm{d}x.$$

(b) Note that $-v_\varepsilon$ solves (4.3.1) with right-hand side $-g$. If $-g \leq 0$ a.e. on $\mathbb{R}^N$, then $(-g)^+ = 0 \in L^1(\mathbb{R}^N)$. By part (a), we deduce that $\varepsilon \int_{\mathbb{R}^N} (-v_\varepsilon)^+ \,\mathrm{d}x \leq 0$, and hence that $-v_\varepsilon \leq 0$ a.e. on $\mathbb{R}^N$. □

We are now ready to prove our main theorem for the elliptic equation (4.3.1).

Proof of Theorem 4.3.1. (a) By the assumptions and Proposition 4.6.8 (d), for every $r > 0$, there exists a unique classical solution $v_{\varepsilon,r} \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ of (4.6.1) satisfying

$$\varepsilon \|D^\alpha v_{\varepsilon,r}\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty} \qquad \text{for all} \qquad \alpha \in \mathbb{N}^N.$$

An Arzelà-Ascoli argument as in the proof of Proposition 4.6.12 (in this case combined with a diagonal extraction argument), shows the existence of classical solutions $v_\varepsilon \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ of (4.3.1) satisfying

$$\varepsilon \|D^\alpha v_\varepsilon\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty} \qquad \text{for all} \qquad \alpha \in \mathbb{N}^N.$$

Moreover, Corollary 4.6.11 ensures that the classical solutions v_ε are unique.

(b) *Existence of L^1-solutions:* Let $\delta > 0$, $g_\delta = g * \omega_\delta \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$, where ω_δ is defined by (4.1.7), and $v_{\varepsilon,\delta} \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ be the solution of (4.3.1) with g_δ as right hand side. By Remark 4.6.1 (a), a difference of solutions is also a solution, and then by Corollary 4.6.14 (b),

$$\varepsilon \|v_{\varepsilon,\delta_1} - v_{\varepsilon,\delta_2}\|_{L^1} \leq \|g_{\delta_1} - g_{\delta_2}\|_{L^1} \qquad \text{for every} \qquad \delta_1, \delta_2 > 0.$$

Hence, $\{v_{\varepsilon,\delta}\}_{\delta>0}$ is Cauchy and there exists $v_\varepsilon \in L^1(\mathbb{R}^N)$ such that $\|v_{\varepsilon,\delta} - v_\varepsilon\|_{L^1} \to 0$ as $\delta \to 0^+$.

Since $v_{\varepsilon,\delta}$ satisfies (4.3.1) with right-hand side g_δ,

$$\varepsilon \int_{\mathbb{R}^N} v_{\varepsilon,\delta}\psi \,\mathrm{d}x - \int_{\mathbb{R}^N} v_{\varepsilon,\delta}\mathcal{L}^\mu[\psi] \,\mathrm{d}x = \int_{\mathbb{R}^N} g_\delta \psi \,\mathrm{d}x \qquad \text{for all} \qquad \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N),$$

and since $v_{\varepsilon,\delta}, g_\delta \to v_\varepsilon, g$ in $L^1(\mathbb{R}^N)$ as $\delta \to 0^+$, we send $\delta \to 0^+$ and find that v_ε is an L^1-distributional solution of (4.3.1).

Uniqueness: Note that $L^1 \subset L^1_{\mathrm{loc}}$. Consider two distributional solutions $v_\varepsilon, \hat{v}_\varepsilon$ of (4.3.1) with right-hand sides $g, \hat{g} \in L^1(\mathbb{R}^N)$. If $g - \hat{g} = 0$ a.e., then $v_\varepsilon - \hat{v}_\varepsilon = B^\mu_\varepsilon[g - \hat{g}] = 0$ by Theorem 4.6.15 (b).

L^1-estimate: By the assumptions, we can take $v_\varepsilon \in L^1(\mathbb{R}^N) \subset L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N)$. Then Theorem 4.6.15 (a) gives

$$\varepsilon\|(v_\varepsilon)^+\|_{L^1} \leq \|(g)^+\|_{L^1}.$$

A similar argument as in the proof of Corollary 4.6.14 concludes the proof.

(c) *Existence of L^∞-solutions:* Proposition 4.6.8 (b) ensures that there exists a unique a.e. solution $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ of

$$\varepsilon v_{\varepsilon,r} - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}] = g,$$

and $\varepsilon\|v_{\varepsilon,r}\|_{L^\infty} \leq \|g\|_{L^\infty}$. Then, by Alaoglu's theorem there exists $\overline{v_\varepsilon} \in L^\infty(\mathbb{R}^N)$ such that, up to a subsequence, $v_{\varepsilon,r_n} \overset{*}{\rightharpoonup} \overline{v_\varepsilon}$ in $L^\infty(\mathbb{R}^N)$ as $r_n \to 0^+$. That is,

$$\lim_{r_n \to 0^+} \int_{\mathbb{R}^N} v_{\varepsilon,r_n}\psi \,\mathrm{d}x = \int_{\mathbb{R}^N} \overline{v_\varepsilon}\psi \,\mathrm{d}x \qquad \text{for all} \qquad \psi \in L^1(\mathbb{R}^N).$$

To finish the existence proof, we need to show that $\overline{v_\varepsilon}$ is in fact a distributional solution of (4.3.1). Consider a function $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$, then $\gamma, \mathcal{L}^{\mu_{r_n}}[\gamma], \mathcal{L}^\mu[\gamma] \in L^1(\mathbb{R}^N)$ (see Lemma 4.3.5 (b)). Since v_{ε,r_n} is a pointwise a.e. solution and $v_{\varepsilon,r_n}, \mathcal{L}^{\mu_{r_n}}[v_{\varepsilon,r_n}] \in L^\infty(\mathbb{R}^N)$, we have by integration and self-adjointness of $\mathcal{L}^{\mu_{r_n}}$ (cf. Lemma 4.3.5 and Remark 4.3.6 (b)) that

$$\varepsilon \int_{\mathbb{R}^N} v_{\varepsilon,r_n}\gamma \,\mathrm{d}x - \int_{\mathbb{R}^N} v_{\varepsilon,r_n}\mathcal{L}^{\mu_{r_n}}[\gamma] \,\mathrm{d}x = \int_{\mathbb{R}^N} g\gamma \,\mathrm{d}x \qquad \text{for all} \qquad \gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N).$$

The weak* L^∞-convergence ensures that

$$\lim_{r_n\to 0^+}\int_{\mathbb{R}^N} v_{\varepsilon,r_n}\gamma\,\mathrm{d}x=\int_{\mathbb{R}^N}\overline{v_\varepsilon}\gamma\,\mathrm{d}x\qquad\text{for all}\qquad \gamma\in C_\mathrm{c}^\infty(\mathbb{R}^N).$$

By Remark 4.6.1 (b), we have that, for any $\gamma\in C_\mathrm{c}^\infty(\mathbb{R}^N)$, $\mathcal{L}^{\mu_{r_n}}[\gamma]\to\mathcal{L}^\mu[\gamma]$ in $L^1(\mathbb{R}^N)$ as $r_n\to 0^+$. Then, since $\|v_{\varepsilon,r_n}\|_{L^\infty}\leq\frac{1}{\varepsilon}\|g\|_{L^\infty}$, we get as $r_n\to 0^+$

$$\int_{\mathbb{R}^N} v_{\varepsilon,r_n}\mathcal{L}^{\mu_{r_n}}[\gamma]\,\mathrm{d}x=\int_{\mathbb{R}^N} v_{\varepsilon,r_n}\mathcal{L}^{\mu}[\gamma]\,\mathrm{d}x+\int_{\mathbb{R}^N} v_{\varepsilon,r_n}(\mathcal{L}^{\mu_{r_n}}-\mathcal{L}^\mu)[\gamma]\,\mathrm{d}x\to\int_{\mathbb{R}^N}\overline{v_\varepsilon}\mathcal{L}^\mu[\gamma]\,\mathrm{d}x,$$

for all $\gamma\in C_\mathrm{c}^\infty(\mathbb{R}^N)$. This shows that

$$\varepsilon\int_{\mathbb{R}^N}\overline{v_\varepsilon}\gamma\,\mathrm{d}x-\int_{\mathbb{R}^N}\overline{v_\varepsilon}\mathcal{L}^\mu[\gamma]\,\mathrm{d}x=\int_{\mathbb{R}^N} g\gamma\,\mathrm{d}x\qquad\text{for all}\qquad\gamma\in C_\mathrm{c}^\infty(\mathbb{R}^N),$$

that is, $\overline{v_\varepsilon}$ is an L^∞-distributional solution of (4.3.1).

Uniqueness: Note that $L^\infty\subset L^1_\mathrm{loc}$. Consider two distributional solutions $v_\varepsilon,\hat{v}_\varepsilon$ of (4.3.1) with right-hand sides $g,\hat{g}\in L^\infty(\mathbb{R}^N)$. If $g-\hat{g}=0$ on $\mathbb{R}^N$, then $v_\varepsilon-\hat{v}_\varepsilon=0$ by Theorem 4.6.15 (b).

L^∞-estimate: Observe that $\pm\frac{1}{\varepsilon}\|g\|_{L^\infty}\in L^\infty(\mathbb{R}^N)\subset L^1_\mathrm{loc}(\mathbb{R}^N)$ are distributional solutions of (4.3.1) with $\pm\|g\|_{L^\infty}$ as right-hand sides. Moreover, $-\|g\|_{L^\infty}\leq g\leq\|g\|_{L^\infty}$. Then Theorem 4.6.15 (b) gives $|v_\varepsilon|\leq\frac{1}{\varepsilon}\|g\|_{L^\infty}$. □

This section is concluded by a proof of the self-adjointness of B^μ_ε.

Proof of Lemma 4.3.4. Let $f_\delta=f*\omega_\delta$ and $g_\delta=g*\omega_\delta$ where ω_δ is defined by (4.1.7). Then $f_\delta\in C_\mathrm{b}^\infty(\mathbb{R}^N)\cap W^{2,1}(\mathbb{R}^N)$ and $g\in C_\mathrm{b}^\infty(\mathbb{R}^N)$, and then by Theorem 4.3.1 (a)–(c), $B^\mu_\varepsilon[f_\delta]\in C_\mathrm{b}^\infty(\mathbb{R}^N)\cap W^{2,1}(\mathbb{R}^N)$, $B^\mu_\varepsilon[g_\delta]\in C_\mathrm{b}^\infty(\mathbb{R}^N)$, and

$$\begin{aligned}\varepsilon B^\mu_\varepsilon[f_\delta]-\mathcal{L}^\mu[B^\mu_\varepsilon[f_\delta]]&=f_\delta(x)\qquad\text{in}\qquad\mathbb{R}^N,\\ \varepsilon B^\mu_\varepsilon[g_\delta]-\mathcal{L}^\mu[B^\mu_\varepsilon[g_\delta]]&=g_\delta(x)\qquad\text{in}\qquad\mathbb{R}^N.\end{aligned}$$

By the regularity and integrability of the terms of the equations (cf. Lemma 4.3.5), we may multiply the first equation by $B^\mu_\varepsilon[g_\delta]$ and the second by $B^\mu_\varepsilon[f_\delta]$, and then integrate both equations in x over $\mathbb{R}^N$. By self-adjointness of $\mathcal{L}^\mu$ (Lemma 4.3.5 (c)), we then find that

$$\begin{aligned}\int_{\mathbb{R}^N} f_\delta B^\mu_\varepsilon[g_\delta]\,\mathrm{d}x &= \int_{\mathbb{R}^N} \big(\varepsilon B^\mu_\varepsilon[f_\delta]-\mathcal{L}^\mu[B^\mu_\varepsilon[f_\delta]]\big)B^\mu_\varepsilon[g_\delta]\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} \big(\varepsilon B^\mu_\varepsilon[g_\delta]-\mathcal{L}^\mu[B^\mu_\varepsilon[g_\delta]]\big)B^\mu_\varepsilon[f_\delta]\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} g_\delta B^\mu_\varepsilon[f_\delta]\,\mathrm{d}x\end{aligned} \tag{4.6.7}$$

To pass to the limit as $\delta\to 0^+$, we first subtract equations to find that

$$\varepsilon B^\mu_\varepsilon[f]-\varepsilon B^\mu_\varepsilon[f_\delta]-\mathcal{L}^\mu\big[B^\mu_\varepsilon[f]-B^\mu_\varepsilon[f_\delta]\big]=f-f_\delta \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N),$$

and hence by Theorem 4.3.1 (b), linearity, and properties of mollifiers,

$$\varepsilon\|B^\mu_\varepsilon[f]-B^\mu_\varepsilon[f_\delta]\|_{L^1}=\varepsilon\|B^\mu_\varepsilon[f-f_\delta]\|_{L^1}\leq\|f-f_\delta\|_{L^1}\to 0 \qquad \text{as} \qquad \delta\to 0^+.$$

On the other hand, by Theorem 4.3.1 (b) and (c), and properties of the mollifiers,

$$\varepsilon\|B^\mu_\varepsilon[f_\delta]\|_{L^1}\leq\|f\|_{L^1},\quad \varepsilon\|B^\mu_\varepsilon[f_\delta]\|_{L^\infty}\leq\|f\|_{L^\infty},\quad \varepsilon\|B^\mu_\varepsilon[g_\delta]\|_{L^\infty}\leq\|g\|_{L^\infty},$$

and $g_\delta\to g$ a.e. Using L^1-convergence for the f-terms and the dominated convergence theorem for the g-terms, we may send $\delta\to 0^+$ in (4.6.7) to get the result. □

4.7 Appendix: Technical results

4.7.1 Proof of Liouville type of theorem

Proof of Theorem 4.3.9. By the definition of distributional solutions,

$$\int_{\mathbb{R}^N} v(y)\mathcal{L}^\mu[\psi](y)\,\mathrm{d}y=0 \qquad \text{for all} \qquad \psi\in C^\infty_\mathrm{c}(\mathbb{R}^N).$$

Let $x\in\mathbb{R}^N$, take $\psi(y)=\omega_\delta(x-y)$, where ω_δ is defined in (4.1.7), and let $v_\delta=v*\omega_\delta\in C_0(\mathbb{R}^N)\cap C^\infty_\mathrm{b}(\mathbb{R}^N)$. By Lemma 4.3.5 (b), $\mathcal{L}^\mu[\psi]\in L^1$, and we may use Fubini's theorem to see that

$$\mathcal{L}^\mu[v_\delta](x)=0 \qquad \text{for every} \qquad x\in\mathbb{R}^N. \tag{4.7.1}$$

Assume that there exists an $\tilde{x}\in\mathbb{R}^N$ such that $v_\delta(\tilde{x})\neq 0$. We only consider the case $v_\delta(\tilde{x})>0$; the proof in the other case is similar. Then $M:=\sup_{x\in\mathbb{R}^N}v_\delta>0$, and since

$v_\delta \in C_0(\mathbb{R}^N)$ there exists an x_0 such that

$$0 < M = \max_{x\in\mathbb{R}^N} v_\delta = v_\delta(x_0).$$

By equation (4.7.1) and Lemma 4.3.5 (b), we then find that

$$\begin{aligned}0 = \mathcal{L}^\mu[v_\delta](x_0) &= \int_{|z|\leq\kappa} v_\delta(x_0+z) - v_\delta(x_0) - z\cdot Dv_\delta(x_0)\,\mathrm{d}\mu(z)\\ &\quad+\int_{|z|>\kappa} v_\delta(x_0+z) - v_\delta(x_0)\,\mathrm{d}\mu(z)\\ &\leq \|D^2 v_\delta\|_{L^\infty(\overline{B}(x_0,\kappa))}\int_{|z|\leq\kappa}|z|^2\,\mathrm{d}\mu(z)\\ &\quad+\int_{|z|>\kappa} v_\delta(x_0+z) - M\,\mathrm{d}\mu(z).\end{aligned}$$

Take any $z_0 \in \operatorname{supp}\mu$. By definition, $z_0 \neq 0$ and $\mu(B(z_0,r)) > 0$ for all $r > 0$. Hence we can take $r, \kappa \in (0,1)$ small enough such that

$$B(z_0,r)\cap\{z\in\mathbb{R}^N : |z|\leq\kappa\} = \emptyset.$$

Since $\kappa < 1$, $v_\delta(x_0+z) - M \leq 0$, and $B(z_0,r)\subset\{z\in\mathbb{R}^N : |z|>\kappa\}$, the above inequality yields that

$$\int_{B(z_0,r)} v_\delta(x_0+z) - M\,\mathrm{d}\mu(z) \geq -\|D^2 v_\delta\|_{L^\infty(\overline{B}(x_0,1))}\int_{|z|\leq1}|z|^2\mathbf{1}_{|z|\leq\kappa}\,\mathrm{d}\mu(z).$$

Taking the limit as $\kappa\to0^+$ using Lebesgue's dominated convergence theorem (the integrand is dominated by $|z|^2$ which is integrable by (A_μ)) gives

$$\int_{B(z_0,r)} v_\delta(x_0+z)\,\mathrm{d}\mu(z) - M\mu(B(z_0,r)) \geq 0.$$

Then by continuity, $v_\delta(x_0+z) = v_\delta(x_0+z_0) + \lambda(|z-z_0|)$ in $B(z_0,r)$ for some modulus of continuity λ, and we find that

$$v_\delta(x_0+z_0) + \lambda(r) \geq \frac{1}{\mu(B(z_0,r))}\int_{B(z_0,r)} v_\delta(x_0+z)\,\mathrm{d}\mu(z) \geq M.$$

Hence, we may send $r\to0^+$ and get that $v_\delta(x_0+z_0)\geq M$. It follows that $v_\delta(x_0+z_0) = M$ since M is the maximum of v_δ.

Repeating the above argument, we find that $v_\delta(x_0+nz_0) = M$ for every $n\in\mathbb{N}$, and thus

$$\limsup_{n\to\infty} v_\delta(x_0+nz_0) \geq M > 0.$$

This is a contradiction since $\lim_{|x|\to\infty} v_\delta(x) = 0$. So, we conclude that $v_\delta(x) = 0$ for every $x \in \mathbb{R}^N$.

By the properties of mollifiers, $v_\delta \to v$ locally uniformly in $\mathbb{R}^N$ as $\delta \to 0^+$, and hence it follows that also $v(x) = 0$ for every $x \in \mathbb{R}^N$. □

4.7.2 Proof of a measure theory result

Proof of Lemma 4.6.4. Remember that we defined

$$M_1(A,B) = \int_A \left(\int_{B-z} \mathrm{d}\nu(x) \right) \mathrm{d}z = \int_A \nu(B-z)\,\mathrm{d}z,$$

$$M_2(A,B) = \int_B \left(\int_{A-z} \mathrm{d}\nu(x) \right) \mathrm{d}z = \int_B \nu(A-z)\,\mathrm{d}z,$$

and that we want to show that $M_1(A,B) = M_2(A,B)$.

Consider the set $C \subset \mathbb{R}^{2N}$ defined as

$$C = \{(x,z) \in \mathbb{R}^{2N} : z \in A, \quad x \in B - z\}.$$

Furthermore, define the sets

$$S = \{x = x_B - x_A : x_A \in A, \quad x_B \in B\} = \bigcup_{x_A \in A} (B - x_A),$$

$$G_x = \{z \in A : x \in B - z\} = \{z \in A : z \in B - x\} = A \cap (B - x).$$

Note that C can also be expressed as

$$C = \{(x,z) \in \mathbb{R}^{2N} : x \in S, \quad z \in G_x\}.$$

Then

$$\begin{aligned} M_1(A,B) &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_A(z)\,\mathbf{1}_{B-z}(x)\,\mathrm{d}\nu(x) \right) \mathrm{d}z = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_C(x,z)\,\mathrm{d}\nu(x) \right) \mathrm{d}z \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_C(x,z)\,\mathrm{d}z \right) \mathrm{d}\nu(x) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_S(x)\,\mathbf{1}_{G_x}(z)\,\mathrm{d}z \right) \mathrm{d}\nu(x) \\ &= \int_S \left(\int_{G_x} \mathrm{d}z \right) \mathrm{d}\nu(x) = \int_S |G_x|\,\mathrm{d}\nu(x), \end{aligned} \tag{4.7.2}$$

where the third equality follows by Tonelli's theorem (the tensor measure is a nonnegative Radon measure), and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$.

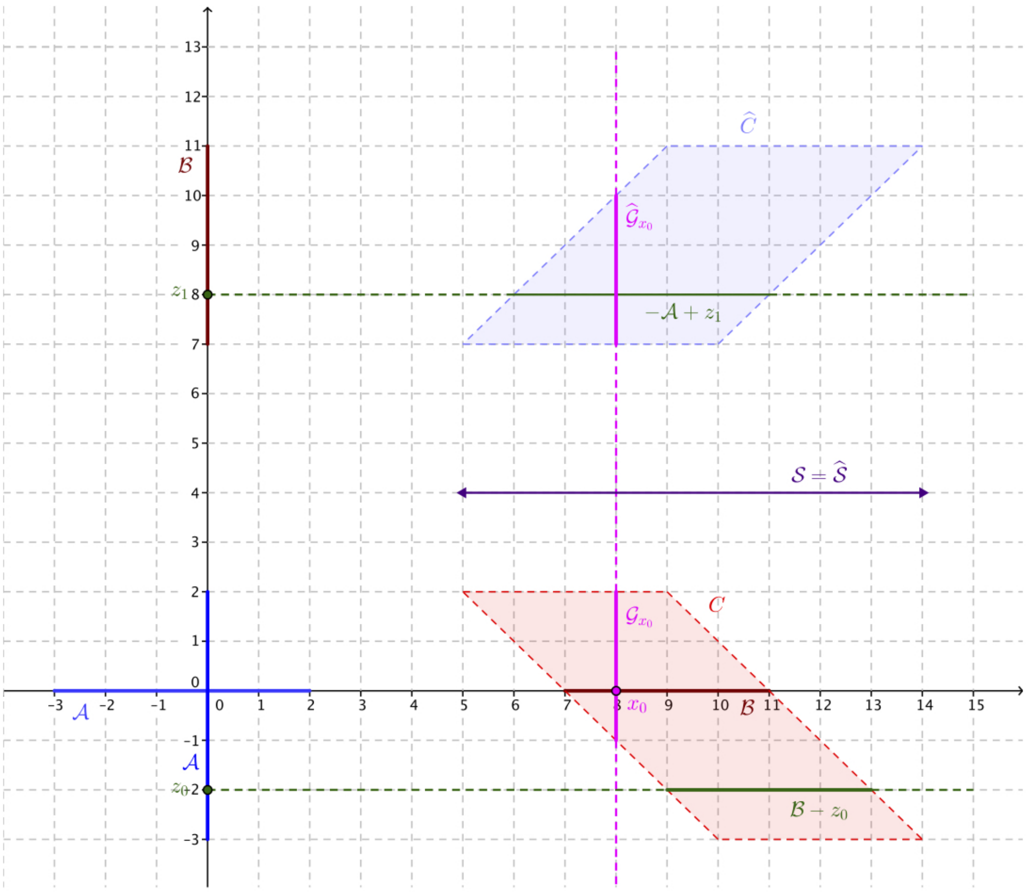

FIGURE 4.1: Geometrical interpretation of Lemma 4.6.4

We can proceed in the same way to change the order of integration in the expression for $M_2(A, B)$, but first we make use of the symmetry of ν

$$M_2(A,B) = \int_B \nu(A-z)\,\mathrm{d}z = \int_B \nu(-A+z)\,\mathrm{d}z = \int_B \left(\int_{z-A} \mathrm{d}\nu(x)\right) \mathrm{d}z.$$

Using the same technique we consider the sets,

$$\widehat{C} = \{(x,z) \in \mathbb{R}^{2N} : z \in B, \quad x \in -A+z\},$$

$$\widehat{S} = \{x = x_B + x_A : x_A \in -A, \quad x_B \in B\} = \bigcup_{x_B \in B} (x_B - A),$$

$$\widehat{G}_x = \{z \in B : x \in -A+z\} = \{z \in B : z \in A+x\} = B \cap (A+x).$$

Second, we follow (4.7.2) to get

$$M_2(A,B) = \int_{\widehat{S}} |\widehat{G}_x|\,\mathrm{d}\nu(x). \tag{4.7.3}$$

Now, note that $S = \widehat{S}$. Moreover, $G_x = A \cap (B - x)$ is just a translation of $\widehat{G}_x = B \cap (A + x)$. Then $|G_x| = |\widehat{G}_x|$, since the Lebesgue measure is invariant under translations. (Consult Figure 4.1 for a visual overview of the sets.)

Finally, we can conclude by (4.7.2) and (4.7.3) that

$$M_1(A, B) = \int_S |G_x| \,\mathrm{d}\nu(x) = \int_{\widehat{S}} |\widehat{G}_x| \,\mathrm{d}\nu(x) = M_2(A, B),$$

which completes the proof. $\square$

Chapter 5

Non-Published research

This chapter is devoted research research works of my thesis that for different reasons have not been published yet, but they have quite an important role in the study of the models developed through this manuscript. There are two directions considered here:

Numerical method for the Fractional Porous Medium Equation when $s \in (0,1)$: In Section 5.1 we formulate a numerical scheme for more general fractional diffusion equations. Important properties of the scheme are presented, but no convergence towards the theoretical solution is proved here. The main reason for not getting convergence in the same way as in Chapter 1 is the lack of regularity of the theoretical solution. A detailed discussion this and other facts and also a possible strategy for proving convergence will also be presented in this section.

An extension problem related to the inverse fractional Laplacian: On the other hand, the results presented in Section 5.2 are consequence of Chapter 2. That chapter study the property of finite or infinite speed of propagation for the equation

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-\frac{\sigma}{2}} u).$$

During the preparation of these results, we decided doing some numerical simulations of the solutions of this model in order to understand how the free boundary was propagated. In this way, we asked ourselves if we could develop an extension theory for the inverse fractional laplacian $(-\Delta)^{-\frac{\sigma}{2}}$ as the one introduced by Caffarelli and Silvestre in [21] for the fractional laplacian $(-\Delta)^{\frac{\sigma}{2}}$.

5.1 Finite difference method for a general fractional porous medium equation

A natural extension of the research presented in Chapter 1 is to find a numerical method for the more general fractional porous medium equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{\frac{\sigma}{2}}(|u|^{m-1}u) = 0, & x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.1.1}$$

for exponents $m \geq 1$, space dimension $N \geq 1$, and fractional exponent $\sigma \in (0,2)$.

We recall that the fractional Laplacian operator $(-\Delta)^s$, $0 < \sigma < 2$, is probably the best known example in the class of nonlocal diffusion operators that are studied because of their interest both in theory and applications, cf. [63, 73]. Indeed, fractional diffusions have a long history in modeling problems in physics, finance, mathematical biology and hydrology. The fractional Laplacian operator is usually defined via Fourier transform for any function f in the Schwartz class as the operator such that

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^\sigma \mathcal{F}(u)(\xi)$$

or via Riesz potential, for a more general class of functions, as

$$(-\Delta)^s f(x) = C_{N,\sigma} \text{P.V.} \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+\sigma}} dy\,, \tag{5.1.2}$$

where $C_{N,\sigma} = 2^{\sigma-1}\sigma\Gamma\left(\frac{N+\sigma}{2}\right)/\pi^{N/2}\Gamma\left(1-\frac{\sigma}{2}\right)$ is a normalization constant. For an equivalence of both formulations see for example [75].

5.1.1 Local formulation of the non-local problem

Our aim is to find numerical approximations for the solutions of the Cauchy Problem for the porous medium equation with fractional diffusion, stated in (5.1.1). We will take $m \geq 1$, $\sigma \in (0,2)$, and the initial function $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and nonnegative. The general theory for existence, uniqueness and regularity of the solution of problem (5.1.1) can be found in [38]. In particular, it is shown that problem (5.1.1) is equivalent to the so-called extension formulation,

$$\begin{cases} L_\sigma[w](x,y,t) = 0, & x \in \mathbb{R}^N,\ y>0,\ t>0, \\ \dfrac{\partial w^{1/m}}{\partial t} = \dfrac{\partial w}{\partial y^\sigma}, & x \in \mathbb{R}^N,\ y=0,\ t>0, \\ w(x,0,0) = f^m(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.1.3}$$

where the extension is defined in terms of the elliptic operator:

$$L_\sigma[v] := \nabla \cdot (y^{1-\sigma}\nabla v)\,,$$

while the *normalized σ-derivative operator* $\dfrac{\partial}{\partial y^\sigma}$ is defined, for any $\sigma \in (0,2)$, as

$$\frac{\partial v}{\partial y^\sigma}(x,0) := \mu_\sigma \lim_{y\to 0} y^{1-\sigma}\frac{\partial v}{\partial y}(x,y), \qquad . \tag{5.1.4}$$

where $\mu_\sigma = 2^{\sigma-1}\Gamma(\sigma/2)/\Gamma(1-\sigma/2)$. The equivalence between (5.1.1) and (5.1.3) holds in the sense of trace and L_σ-harmonic extension operators, that is,

$$u(x,t) = Tr(w^{1/m}(x,y,t)), \quad w(x,y,t) = E_\sigma(u^m(x,t)).$$

Note 5.1.1. For a smooth enough function v, the extension operator L_σ defined in (5.1.4) can also be written as

$$L_\sigma[v](x,y) = y^{1-\sigma}\Delta_{x,y}v(x,y) + (1-\sigma)y^{-\sigma}\frac{\partial v}{\partial y}(x,y)\,,$$

where $\Delta_{x,y}$ is the $N+1$ dimensional Laplacian operator.

5.1.2 Extension problem in a bounded domain

In order to construct a numerical solution to Problem (5.1.3), we perform a monotone approximation of the solution in the whole space by the solutions of the problem posed in a bounded domain.

We consider positive numbers $X_1, \dots, X_N, Y, T$. We define the bounded domain $\Omega = (-X_1, X_1) \times \ldots \times (-X_N, X_N) \times (0, Y)$, and set $\Gamma = \partial\Omega$. For convenience we also divide the boundary in two parts (see Figure 1.1):

$$\Gamma_d = [-X_1, X_1] \times \ldots \times [-X_N, X_N] \times \{0\}$$

(the base), and $\Gamma_h = \partial\Omega\backslash\Gamma_d$ (the lateral boundary of the extended domain). With these notations, we formulate the corresponding problem in the bounded domain as

$$\begin{cases} L_\sigma[w](x,y,t) = 0, & (x,y) \in \Omega,\ t \in (0,T], \\ \dfrac{\partial w^{1/m}}{\partial t}(x,0,t) = \dfrac{\partial w}{\partial y^\sigma}(x,0,t), & (x,y) \in \Gamma_d,\ t \in (0,T], \\ w(x,0,0) = f^m(x), & (x,y) \in \Gamma_d, \\ w(x,y,t) = 0, & (x,y) \in \Gamma_h\,. \end{cases} \tag{5.1.5}$$

Notice that we have imposed homogeneous boundary conditions on Γ_h.

In the sequel, we will consider the problem with $N = 1$ in order to simplify de notation, but all the arguments are also valid for $N > 1$ without much effort.

5.1.3 Discretization of the σ-derivative

Given the σ-harmonic extension problem

$$\begin{cases} L_\sigma[v](x,y) = \nabla\cdot(y^{1-\sigma}\nabla v) = 0 & x \in \mathbb{R}^N,\ y > 0, \\ v(x,0) = g(x) & x \in \mathbb{R}^N, \end{cases} \tag{5.1.6}$$

the explicit solution is given by a convolution of the boundary condition with the kernel

$$P(x,y) = d_{N,\sigma}\frac{y^\sigma}{(|x|^2+y^2)^{\frac{N+\sigma}{2}}}, \tag{5.1.7}$$

that is,

$$v(x,y) = \int_{\mathbb{R}^N} P(x-\xi,y)g(\xi)d\xi. \tag{5.1.8}$$

In view of this, we introduce the discretized σ-derivative at $y = 0$ as follows:

$$F(x,y) := \sigma\frac{v(x,y)-v(x,0)}{y^\sigma} = \sigma d_{N,\sigma}\int_{\mathbb{R}^N}\frac{g(\xi)-g(x)}{(|x-\xi|^2+y^2)^{\frac{N+\sigma}{2}}}d\xi\,. \tag{5.1.9}$$

With this definition, we have

$$\lim_{y\to 0} F(x,y) = \lim_{y\to 0} y^{1-\sigma}\frac{\partial v}{\partial y}(x,y). \tag{5.1.10}$$

Notice that the operator $\frac{\partial}{\partial y^\sigma}$ satisfies

$$\frac{\partial v}{\partial y^\sigma}(x,0) := \mu_\sigma \lim_{y\to 0} y^{1-\sigma}\frac{\partial v}{\partial y}(x,y) = -(-\Delta)^s g(x).$$

Summing up, it seems that $\mu_\sigma F(x,y)$ could be a good candidate to be used as the discretization of $\frac{\partial v}{\partial y^\sigma}(x,0)$.

We are interested in the order of the discretization. For that we have to compute the difference

$$\begin{aligned}\mu_\sigma F(x,y) - \frac{\partial v}{\partial y^\sigma}(x,0) &= \mu_\sigma \sigma d_{N,\sigma} \int_{\mathbb{R}^N} \frac{g(\xi)-g(x)}{(|x-\xi|^2+y^2)^{\frac{N+\sigma}{2}}} d\xi + (-\Delta)^s g(x) \\ &= C_{N,\sigma} \int_{\mathbb{R}^N} [g(\xi)-g(x)] \left(\frac{1}{(|x-\xi|^2+y^2)^{\frac{N+\sigma}{2}}} - \frac{1}{|x-\xi|^{N+\sigma}} \right) d\xi\end{aligned}$$

Theorem 5.1.1. *Consider F and $\dfrac{\partial v}{\partial y^\sigma}$ as above, and $g \in C_b^2(\mathbb{R}^N)$. Then,*

$$|\mu_\sigma F(x,y) - \frac{\partial v}{\partial y^\sigma}(x,0)| \le O(y^{2-\sigma}),$$

that is,

$$|\mu_\sigma \sigma \frac{v(x,y)-v(x,0)}{y^\sigma} - \frac{\partial v}{\partial y^\sigma}(x,0)| \le O(y^{2-\sigma}).$$

Proof. We want to have an estimate in terms of y of the expression

$$\begin{aligned} I &= \frac{1}{C_{N,\sigma}} \left(\mu_\sigma F(x,y) - \frac{\partial v}{\partial y^\sigma}(x,0) \right) \\ &= \int_{\mathbb{R}^N} [g(\xi)-g(x)] \left(\frac{1}{(|x-\xi|^2+y^2)^{\frac{N+\sigma}{2}}} - \frac{1}{|x-\xi|^{N+\sigma}} \right) d\xi \end{aligned}$$

We split the integral into $I_R + I_C$, where I_R is the above integral computed in $B_R(x)$ and I_C the integral computed in $B_R^c := \mathbb{R}^N \backslash B_R(x)$. For simplicity, we are going to estimate all the integrals when $x=0$, but the calculation for a general $x \in \mathbb{R}^N$ is analogous.

Let us first estimate the integral outside the origin, I_C:

$$\begin{aligned} |I_C| &\le 2||g||_\infty \int_{B_R^c} \frac{1}{|\xi|^{N+\sigma}} - \frac{1}{(|\xi|^2+y^2)^{\frac{N+\sigma}{2}}} d\xi \\ &= \frac{C_g}{y^\sigma} \int_{B_{R/y}^c} \frac{(|z|^2+1)^{\frac{N+\sigma}{2}} - |z|^{N+\sigma}}{|z|^{N+\sigma}(|z|^2+1)^{\frac{N+\sigma}{2}}} dz \\ &= \frac{C_{g,N}}{y^\sigma} \int_{R/y}^\infty \frac{(r^2+1)^{\frac{N+\sigma}{2}} - r^{N+\sigma}}{r^{\sigma+1}(r^2+1)^{\frac{N+\sigma}{2}}} dr = \frac{C_{g,N}}{y^\sigma} \int_{R/y}^\infty \frac{f(1)-f(0)}{r^{\sigma+1}(r^2+1)^{\frac{N+\sigma}{2}}} dr. \end{aligned}$$

We first use the change of variables $\xi = yz$ and then a change to polar coordinates. We also define, for a fixed r the function

$$f(x) = (r^2+x)^{\frac{N+\sigma}{2}},$$

with derivative

$$f'(x) = \frac{N+\sigma}{2}(r^2+x)^{\frac{N+\sigma-2}{2}}.$$

Then $f(1) - f(0) = f'(\eta)$ for some $\eta \in [0,1]$. Now we need to consider two separate cases.

• If $\mathbf{N} + \sigma \geq \mathbf{2}$, then,

$$f'(x) \leq \frac{N+\sigma}{2}(r^2+1)^{\frac{N+\sigma-2}{2}} \quad \forall x \in [0,1].$$

In this way,

$$\begin{aligned} |I_C| &\leq \frac{C_{g,N,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{(r^2+1)^{\frac{N+\sigma-2}{2}}}{r^{\sigma+1}(r^2+1)^{\frac{N+\sigma}{2}}}dr = \frac{C_{g,N,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{1}{r^{\sigma+1}(r^2+1)}dr \\ &\leq \frac{C_{g,N,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{1}{r^{\sigma+3}}dr = \frac{C_{g,N,\sigma}}{y^\sigma}\left(\frac{y}{R}\right)^{\sigma+2} = \frac{C_{g,N,\sigma}}{R^{\sigma+2}}y^2. \end{aligned}$$

• If $\mathbf{N} + \sigma < \mathbf{2}$, that is, $N = 1$ and $\sigma < 1$ then,

$$f'(x) \leq \frac{1+\sigma}{2}r^{\sigma-1} \quad \forall x \in [0,1].$$

In this way,

$$\begin{aligned} |I_C| &\leq \frac{C_{g,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{r^{\sigma-1}}{r^{\sigma+1}(r^2+1)^{\frac{1+\sigma}{2}}}dr = \frac{C_{g,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{1}{r^2(r^2+1)^{\frac{1+\sigma}{2}}}dr \\ &\leq \frac{C_{g,\sigma}}{y^\sigma}\int_{R/y}^\infty \frac{1}{r^{\sigma+3}}dr = \frac{C_{g,\sigma}}{R^{\sigma+2}}y^2. \end{aligned}$$

This means that for every $N \geq 1$ and $\sigma \in (0,2)$,

$$|I_C| \leq \frac{C_{g,N,\sigma}}{R^{\sigma+2}}y^2. \tag{5.1.11}$$

And now, we estimate the integral inside de ball of radius R, I_R: since $\frac{\nabla g(0)\cdot\xi}{(|\xi|^2+y^2)^{\frac{N+\sigma}{2}}}$ and $\frac{\nabla g(0)\cdot\xi}{|\xi|^{N+\sigma}}$ are both odd functions with respect to ξ, they integrate zero in any ball centered in the origin. In this way,

$$\begin{aligned} |I_B| &= \left|\int_{B_R}[g(\xi) - g(0) - \nabla g(0)\cdot\xi]\left(\frac{1}{(|\xi|^2+y^2)^{\frac{N+\sigma}{2}}} - \frac{1}{|\xi|^{N+\sigma}}\right)d\xi\right| \\ &\leq ||D^2 g||_\infty \int_{B_R}|\xi|^2\left(\frac{1}{|\xi|^{N+\sigma}} - \frac{1}{(|\xi|^2+y^2)^{\frac{N+\sigma}{2}}}\right)d\xi \\ &= C_{g,N}\int_0^R r^{1-\sigma}\frac{(r^2+y^2)^{\frac{N+\sigma}{2}} - r^{N+\sigma}}{(r^2+y^2)^{\frac{N+\sigma}{2}}}dr. \end{aligned}$$

With the same trick with the function f defined above, we need to split the calculation in two cases again.

• If $\mathbf{N}+\sigma \geq \mathbf{2}$, then,

$$\begin{aligned} |I_B| &\leq C_{g,N,\sigma}\int_0^R r^{1-\sigma}\frac{(r^2+y^2)^{\frac{N+\sigma-2}{2}}y^2}{(r^2+y^2)^{\frac{N+\sigma}{2}}}dr = C_{g,N,\sigma}y^2\int_0^R r^{1-\sigma}\frac{1}{r^2+y^2}dr \\ &= C_{g,N,\sigma}y^2\int_0^{R/y} y^{1-\sigma}s^{1-\sigma}\frac{y}{y^2(s^2+1)}ds = C_{g,N,\sigma}y^{2-\sigma}\int_0^{R/y}\frac{s^{1-\sigma}}{(s^2+1)}ds. \\ &\leq C_{g,N,\sigma}y^{2-\sigma}\int_0^{\infty}\frac{s^{1-\sigma}}{(s^2+1)}ds = d_{g,N,\sigma}y^{2-\sigma}, \end{aligned}$$

where we have used the change of variables $r = sy$ and the integrability of the function $\frac{s^{1-\sigma}}{(s^2+1)}$ for every $\sigma \in (0,2)$.

• If $\mathbf{N}+\sigma < \mathbf{2}$, that is, $N=1$ and $\sigma < 1$ then,

$$\begin{aligned} |I_B| &\leq C_g\int_0^R r^{1-\sigma}\frac{(r^2+y^2)^{\frac{1+\sigma}{2}}-r^{1+\sigma}}{(r^2+y^2)^{\frac{1+\sigma}{2}}}dr = C_{g,\sigma}\int_0^R r^{1-\sigma}\frac{r^{1+\sigma-2}y^2}{(r^2+y^2)^{\frac{1+\sigma}{2}}}dr \\ &= C_{g,\sigma}y^2\int_0^R\frac{1}{(r^2+y^2)^{\frac{1+\sigma}{2}}}dr = C_{g,\sigma}y^2\int_0^{R/y}\frac{y}{y^{1+\sigma}(s^2+1)^{\frac{1+\sigma}{2}}}ds \\ &= C_{g,\sigma}y^{2-\sigma}\int_0^{R/y}\frac{1}{(s^2+1)^{\frac{1+\sigma}{2}}}ds = C_{g,\sigma}y^{2-\sigma}\int_0^{\infty}\frac{1}{(s^2+1)^{\frac{1+\sigma}{2}}}ds \\ &= d_{g,\sigma}y^{2-\sigma} \end{aligned}$$

where again we have used the change $r = sy$ and the integrability of the function $\frac{1}{(s^2+1)^{\frac{1+\sigma}{2}}}$ for every $\sigma > 0$. □

Note 5.1.2. Notice that the case $\sigma \neq 1$ generalizes the usual result of the order of discretization of the forward Euler discretization for a first derivative.

5.1.4 Discrete formulation

In order to solve problem (5.1.5) for $t \in [0,T]$, we first perform a time and space discretization (see Figure 1.2 for a representation of this discretization). For the time discretization we choose J uniformly spaced steps, and then $\Delta t = T/J$ and

$$0 \leq j\Delta t \leq T, \quad j = 0,...,J, \qquad t_j = j\Delta t.$$

We also need to discretize the space domain $\overline{\Omega} = [-X, X] \times [0, Y]$. Let I, K be the number of steps on each space direction,

$$0 \leq i\Delta x \leq 2X, \quad i = 0, ..., I \text{ where } \Delta x = 2X/I \text{ and } x_i = i\Delta x - X,$$

$$0 \leq k\Delta y \leq Y, \quad k = 0, ..., K \text{ where } \Delta y = Y/K \text{ and } y_k = k\Delta y.$$

We use the notation

$$w(x_i, y_k, t_j) = (w_j)_i^k \tag{5.1.12}$$

for the values of the theoretical solution w to Problem (5.1.5) in the points of the mesh, and

$$w(x_i, y_k, t_j) \approx (W_j)_i^k \tag{5.1.13}$$

for the solution of the numerical method.

5.1.5 Numerical method

We recall that in problem (5.1.5) we are dealing with he following extension operator,

$$L_\sigma[v](x, y) := \nabla \cdot (y^{1-\sigma} \nabla v) = y^{1-\sigma} \Delta_{x,y} v + (1-\sigma) y^{-\sigma} \frac{\partial v}{\partial y}.$$

For $N = 1$ and $\Delta y = \Delta x$, we propose the discretized version of L_σ given by,

$$L_\sigma^D \left[v_i^k \right] := y_k^{1-\sigma} \frac{v_{i+1}^k + v_{i-1}^k + v_i^{k+1} + v_i^{k-1} - 4v_i^k}{\Delta x^2} + \frac{(1-\sigma)}{y_k^\sigma} \frac{v_i^{k+1} - v_i^{k-1}}{2\Delta x}. \tag{5.1.14}$$

where we have use the notation $v_i^k := v(x_i, y_k)$. In this way, we formulate following the numerical method in order to try to approximate the solutions of (5.1.5): for each time step $j = 1, \ldots, J$, we solve the linear system of equations

$$\begin{cases} L_\sigma^D \left[(W_j)_i^k \right] = 0, & 0 < i < I, 0 < k < K, \\ (W_j)_i^0 = \left(\nu_\sigma \dfrac{\Delta t}{\Delta x^\sigma} \left((W_{j-1})_i^1 - (W_{j-1})_i^0 \right) + [(W_{j-1})_i^0]^{1/m} \right)^m, & \text{if } 0 < i < I, \\ (W_j)_i^k = 0, & \text{on the } \Gamma_h \text{ nodes.} \end{cases} \tag{5.1.15}$$

We recall that $\nu_\sigma = \sigma\mu_\sigma$ and note that the second equation is explicit in the sense that all the terms in the right-hand side of the equation are known from the previous step.

To start the numerical method, we use the solution of

$$\begin{cases} L_\sigma^D \left[(W_0)_i^k \right] = 0, & 0 < i < I, 0 < k < K, \\ (W_0)_i^0 = f^m(x_i), & \text{if } 0 < i < I, \\ (W_0)_i^k = 0, & \text{on the } \Gamma_h \text{ nodes.} \end{cases} \tag{5.1.16}$$

5.1.6 Local truncation error: a partial result

We define the local truncation error $(\tau_j)_i^k$ as the error that comes from plugging the solution w to Problem (5.1.5) into the numerical method (5.1.15)-(5.1.16). Let us also write

$$\Lambda = \max_{i,k,j} |(\tau_j)_i^k|. \tag{5.1.17}$$

In the following theorem we will assume that the initial condition f has enough regularity in order to have the best regularity expected for the solution w of problem

Theorem 5.1.2. *Let w be the solution to Problem (5.1.5) with $m \geq 1$ and f regular enough. Then,*

$$\max_{i,j} \left\{ \left| (\tau_j)_i^0 \right| \right\} = O(\Delta t(\Delta x^{2-\sigma} + \Delta t)). \tag{5.1.18}$$

Proof. Of course, the local truncation error in the boundary nodes situated on the part Γ_h of the boundary is zero since we have imposed that the solution is zero in Γ_h and equal to $f^m(x)$ in Γ_d at time $t = 0$ as in Problem (5.1.5).

If $0 < i < I$ and $k = 0$ (i.e., at the boundary nodes Γ_h), the local truncation error is calculated as

$$\begin{aligned} (\tau_{j-1})_i^0 :&= \nu_\sigma \frac{\Delta t}{\Delta x^\sigma} [(w_{j-1})_i^1 - (w_{j-1})_i^0] + [(w_{j-1})_i^0]^{1/m} - [(w_j)_i^0]^{1/m} \\ &= \Delta t \big[\frac{\partial w}{\partial y^\sigma}(x_i, 0, t_{j-1}) + O(\Delta x^{2-\sigma}) \big] - \Delta t \big[\frac{\partial w^{1/m}}{\partial t}(x_i, 0, t_{j-1}) + O(\Delta t) \big] \\ &= O(\Delta t \Delta x^{2-\sigma}) + O(\Delta t^2) = O(\Delta t(\Delta t + \Delta x^{2-\sigma})). \end{aligned}$$

We do not need any proof regularity in the discretization of term involving $\frac{\partial w}{\partial y^\sigma}$ since the result is consequence of Theorem 5.1.1. The regularity required for the time discretization of $\frac{\partial w^{1/m}}{\partial t}(x_i, 0, t_{j-1})$ is obtained by observing that $\frac{\partial w^{1/m}}{\partial t}(x_i, 0, t_{j-1}) = \frac{\partial u}{\partial t}(x_i, t_{j-1})$ where u is the solution of (5.1.1) and the time regularity is inherited from f. □

We would also like to have an estimate of the local truncation error in the interior nodes as we obtained in Chapter 1 when $\sigma = 1$. This is not trivially obtained from the regularity of the solution of (5.1.5), since higher order regularity in the y-direction is not true anymore. To be more specific, we would like to say (This is not true!) that $w \in C^\infty(\mathbb{R}^N)$ and then, for any interior node we will have that

$$
\begin{aligned}
(\tau_j)_i^j &= y_k^{1-\sigma} \frac{w_{i+1}^k + w_{i-1}^k + w_i^{k+1} + w_i^{k-1} - 4w_i^k}{\Delta x^2} + \frac{(1-\sigma)}{y_k^\sigma} \frac{w_i^{k+1} - w_i^{k-1}}{2\Delta x} \\
&= y_k^{1-\sigma} \left(\Delta_{x,y} w + O(\Delta x^2) \right) + \frac{(1-\sigma)}{y_k^\sigma} \left(\frac{\partial w}{\partial y} + O(\Delta x^2) \right).
\end{aligned}
$$

Then, since $y_k = k\Delta x$, we use the first equation of (5.1.5) to get that

$$
\left| (\tau_j)_i^j \right| \leq y_k^{1-\sigma} O(\Delta x^2) + \frac{(1-\sigma)}{y_k^\sigma} O(\Delta x^2) = O(\Delta x^{2-\sigma}).
$$

But us we said at the beginning of this discussion, this kind of higher order regularity is not true. Let us give at least an idea of why: The first equation in (5.1.5) says that for any $(x, y) \in \Omega$ we have that

$$
y^{1-\sigma} \frac{\partial^2 v}{\partial y^2} = -y^{1-\sigma} \Delta_x v - (1-\sigma) y^{-\sigma} \frac{\partial v}{\partial y},
$$

that is,

$$
\frac{\partial^2 v}{\partial y^2} = -\Delta_x v - (1-\sigma) y^{-1} \frac{\partial v}{\partial y}. \tag{5.1.19}
$$

On the other hand we know that $\frac{\partial v}{\partial y^\sigma} = \lim_{y \to 0^+} y^{1-\sigma} \frac{\partial v}{\partial y} < \infty$ which means that

$$
\frac{\partial v}{\partial y} \sim y^{\sigma - 1} \qquad \text{as} \qquad y \to 0^+.
$$

Using the above estimate on (5.1.19) we get that

$$
\frac{\partial^2 v}{\partial y^2} \sim y^{\sigma-2} \longrightarrow \infty \qquad \text{as} \qquad y \to 0^+.
$$

So not even regularity of order two in the y-direction is expected for the solution of (5.1.5).

Note 5.1.3. The lack of regularity in the y-direction is not the end of all hope of obtaining convergence of the numerical method. See for example Theorem 5.1.1 where no regularity of the solution is required to obtain a good discretization of the operator $\frac{\partial v}{\partial y^\sigma}$ and instead of this, we make use of the particular form of the explicit solution of the σ-extension problem.

5.1.7 Properties of the scheme: maximum principle, existence and uniqueness of solutions

The quantity defined bellow

$$b_{max} := \max_x\{f^m(x)\} = \|f\|_\infty^m$$

will be used in what follows.

Theorem 5.1.3 (Discrete maximum principle)**.** *Let $(W_j)_i^k$ be the solution of the numerical method (5.1.15)-(5.1.16) with $m \geq 1$ and bounded non-negative f. Assume that*

$$\Delta t \leq C(m,f)\Delta x^\sigma, \quad \textit{where} \quad C(m,f) = [m(b_{max})^{\frac{m-1}{m}}\nu_\sigma]^{-1}. \tag{5.1.20}$$

Then for every i,k,j we have

$$0 \leq (W_j)_i^k \leq b_{max}.$$

We start by proving the following lemma.

Lemma 5.1.4. *Let $(W_j)_i^k$ be the solution to the numerical method (5.1.15)-(5.1.16) with $m \geq 1$ and bounded non-negative f. Then, for a fixed time $t_j = j\Delta t$, we have that*

$$\max_{i,k}\{(W_j)_i^k\} = \max_i\{(W_j)_i^0\},$$

that is, for a fixed time the maximum is always attained at the boundary Γ_d.

Proof. Since we are working on a fixed time, we will skip the subindex j and write W_i^k instead of $(W_j)_i^k$. We consider the discretization defined in (5.1.14) defined by

$$L_\sigma^D\left[W_i^k\right] := y_k^{1-\sigma}\frac{W_{i+1}^k + W_{i-1}^k + W_i^{k+1} + W_i^{k-1} - 4W_i^k}{\Delta x^2} + \frac{(1-\sigma)}{y_k^\sigma}\frac{W_i^{k+1} - W_i^{k-1}}{\Delta x}.$$

Assume that there exists an interior node (i,k) such that

$$W_i^k = \max_{a,b}\{W_a^b\},$$

that is, the maximum is attained there. Note that, since we are talking about an interior node, $k \geq 1$ and we can write $y_k = k\Delta x$. Then, equation $L_\sigma^D\left[W_i^k\right] = 0$ obtained from (5.1.14) becomes,

$$k^{1-\sigma}\frac{W_{i+1}^k + W_{i-1}^k + W_i^{k+1} + W_i^{k-1} - 4W_i^k}{\Delta x^{1+\sigma}} + \frac{(1-\sigma)}{k^\sigma}\frac{W_i^{k+1} - W_i^{k-1}}{\Delta x^{1+\sigma}} = 0.$$

We recall that $W_i^{k+1}, W_i^{k-1}, W_{i+1}^k, W_{i+1}^k \leq W_i^k$. In this way, if $\sigma \in (0,1]$, we have that

$$\begin{aligned}
0 &\leq k^{1-\sigma}\frac{3W_i^k + W_i^{k-1} - 4W_i^k}{\Delta x^{1+\sigma}} + \frac{(1-\sigma)}{k^\sigma}\frac{W_i^k - W_i^{k-1}}{\Delta x^{1+\sigma}} \\
&= \frac{k}{k^\sigma}\frac{W_i^{k-1} - W_i^k}{\Delta x^{1+\sigma}} + \frac{(1-\sigma)}{k^\sigma}\frac{W_i^k - W_i^{k-1}}{\Delta x^{1+\sigma}} \\
&= (k+\sigma-1)\frac{W_i^{k-1} - W_i^k}{k^\sigma \Delta x^{1+\sigma}}.
\end{aligned}$$

We know that $k+\sigma-1 \geq 0$ and then $W_i^{k-1} \geq W_i^k$. Even more, since W_i^k is the maximum, we have that $W_i^{k-1} = W_i^k$. Proceeding by induction on k, we get that

$$W_i^k = W_i^{k-1} = W_i^{k-2} = \ldots = W_i^0,$$

that is, the maximum is attained at Γ_d. On the other hand, if $\sigma \in (1,2)$, we have that

$$\begin{aligned}
0 &\leq k^{1-\sigma}\frac{3W_i^k + W_i^{k+1} - 4W_i^k}{\Delta x^{1+\sigma}} + \frac{(\sigma-1)}{k^\sigma}\frac{W_i^{k-1} - W_i^{k+1}}{\Delta x^{1+\sigma}} \\
&\leq \frac{k}{k^\sigma}\frac{W_i^{k+1} - W_i^k}{\Delta x^{1+\sigma}} + \frac{(\sigma-1)}{k^\sigma}\frac{W_i^k - W_i^{k+1}}{\Delta x^{1+\sigma}} \\
&\leq [k+1-\sigma]\frac{W_i^{k+1} - W_i^k}{k^\sigma \Delta x^{1+\sigma}}.
\end{aligned} \tag{5.1.21}$$

Again, since $k+1-\sigma \geq 0$ we get that $W_i^{k+1} = W_i^k$ and as consequence,

$$W_i^k = W_i^{k+1} = W_i^{k+2} = \ldots = W_i^K,$$

which is a contradiction with the fact that $W_i^K = 0$ by assumption. □

Proof of Theorem 5.1.3. Using the result of Lemma 5.1.4, we only need to prove the maximum principle at the nodes of the boundary Γ_d. We will do the proof by induction on each time step. By Lemma 5.1.4, we know that

$$0 \leq (W_0)_i^k \leq b_{max}.$$

Now we assume that

$$0 \leq (W_{j-1})_i^k \leq b_{max},$$

then,

$$[(W_j)_i^0]^{1/m} = \nu_\sigma \frac{\Delta t}{\Delta x^\sigma}\big((W_{j-1})_i^1 - (W_{j-1})_i^0\big) + [(W_{j-1})_i^0]^{1/m}.$$

Now, we change variables to $(U_j)_i^k = [(W_j)_i^k]^{1/m}$ and use Mean Value Theorem to obtain, for some $\xi \in [(U_{j-1})_i^1, (U_{j-1})_i^0]$, that

$$(U_j)_i^0 = \nu_\sigma m \xi^{m-1} \frac{\Delta t}{\Delta x^\sigma} (U_{j-1})_i^1 + \left[1 - \nu_\sigma m \xi^{m-1} \frac{\Delta t}{\Delta x^\sigma}\right] (U_{j-1})_i^0. \tag{5.1.22}$$

Note that for the new variables, we have that $0 \leq (U_0)_i^k \leq (b_{max})^{1/m}$. At this point , thanks to our induction hypothesis and the value of the constant defined in (5.1.20), it follows that $\nu_\sigma m \xi^{m-1} \dfrac{\Delta t}{\Delta x^\sigma} \leq 1$ and therefore

$$|(U_j)_i^0| \leq \varphi'(\xi) \nu_\sigma \frac{\Delta t}{\Delta x^\sigma} (b_{max})^{1/m} + \left[1 - \varphi'(\xi) \nu_\sigma \frac{\Delta t}{\Delta x^\sigma}\right] (b_{max})^{1/m} = (b_{max})^{1/m}. \tag{5.1.23}$$

The same argument holds for $(U_j)_i^0 \geq 0$. □

Corollary 5.1.5. *If $\Delta t \leq C(m, f) \Delta x^\sigma$, then Problem (5.1.15)-(5.1.16) has a unique solution.*

We are a bit sketchy with these rather standard proofs of Theorem 5.1.3 and Corollary 5.1.5, the reader can consult a more detailed exposition on Chapter 1 or in the paper [40] where the case $\sigma = 1$ is covered. Basically, two solutions $(V_j)_i^k$ and $(W_j)_i^k$ with the same initial condition are considered. Then $(Z_j)_i^k = (V_j)_i^k - (W_j)_i^k$ is also a solution with zero initial condition. By the maximum principle we have $(Z_j)_i^k = 0$, hence $(V_j)_i^k = (W_j)_i^k$. Since, for a linear system of equations with the same number of unknowns and equations, existence is equivalent to uniqueness, the required result is proved.

5.2 An extension problem related to the inverse fractional Laplacian

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a smooth and bounded function. Consider for $\sigma \in (0, 2)$ the following σ-extension problem to the upper half space with Neumann boundary conditions:

$$\begin{cases} \nabla \cdot \left(y^{1-\sigma} \nabla w\right) = 0 & x \in \mathbb{R}^N, \ y > 0, \\ \dfrac{\partial w}{\partial y^\sigma}(x, 0) = -f(x) & x \in \mathbb{R}^N, \end{cases} \tag{5.2.1}$$

where $\frac{\partial w}{\partial y^\sigma}$ is defined as in (5.1.10). We have the following result which shows that the inverse fractional laplacian $(-\Delta)^{-\frac{\sigma}{2}}$ defined as in Chapter 2 is a Neumann to Dirichlet operator for problem (5.2.1).

Theorem 5.2.1. *Let w be the solution of (5.2.1) for $\sigma \in (0, 2)$ ($\sigma \in (0, 1)$ if $N = 1$) and a smooth and bounded boundary data $f : \mathbb{R}^N \to \mathbb{R}$. Then, for some constant $\tilde{d}_{N,\sigma}$,* *we have that*

$$\lim_{y\to 0^+} w(x,y) = \tilde{d}_{N,\sigma}(-\Delta)^{-\frac{\sigma}{2}} f(x). \tag{5.2.2}$$

We prove first the following lemma in order to simplify the understanding and the ideas of the proof.

Lemma 5.2.2. *Let w be the solution of (5.2.1) for $\sigma \in (0,2)$. Let also $f : \mathbb{R}^N \to \mathbb{R}$ be smooth and bounded. Consider the function $z(x,y) = y^{1-\sigma}\frac{\partial w}{\partial y}(x,y)$. Then z solves the following σ-harmonic extension problem:*

$$\begin{cases} \nabla\cdot\left(y^{\sigma-1}\nabla z\right) = 0 & x\in\mathbb{R}^N,\ y>0, \\ z(x,0) = -f(x) & x\in\mathbb{R}^N, \end{cases} \tag{5.2.3}$$

Proof. We recall that the first equation of (5.2.1) is equivalent to say that for every $(x,y) \in \mathbb{R}^N \times (0,\infty)$ we have that

$$y^{1-\sigma}\frac{\partial^2 w}{\partial y^2} + y^{1-\sigma}\Delta_x w + (1-\sigma)y^{-\sigma}\frac{\partial w}{\partial y} = 0.$$

Then, using the above identity, a simple computation shows that

$$\nabla z = \begin{bmatrix} y^{1-\sigma}\dfrac{\partial}{\partial y}\left(\nabla_x w\right) \\ (1-\sigma)y^{-\sigma}\dfrac{\partial w}{\partial y} + y^{1-\sigma}\dfrac{\partial^2 w}{\partial y^2} \end{bmatrix} = \begin{bmatrix} y^{1-\sigma}\dfrac{\partial}{\partial y}\left(\nabla_x w\right) \\ -y^{1-\sigma}\Delta_x w \end{bmatrix}.$$

In this way,

$$\begin{aligned} \nabla\cdot\left(y^{\sigma-1}\nabla z\right) &= \nabla\cdot\begin{bmatrix} \dfrac{\partial}{\partial y}\left(\nabla_x w\right) \\ -\Delta_x w \end{bmatrix} \\ &= \frac{\partial}{\partial y}\left(\Delta_x w\right) - \frac{\partial}{\partial y}\left(\Delta_x w\right) = 0. \end{aligned}$$

Moreover, by definition, $z(x,y) = y^{1-\sigma}\frac{\partial w}{\partial y}(x,y)$ and we can use the second equation of (5.2.1) to show that

$$\begin{aligned} z(x,0) &= \lim_{y\to 0^+} z(x,y) \\ &= \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial w}{\partial y}(x,y) \\ &= \frac{\partial w}{\partial y^\sigma}(x,0) = -f(x) \end{aligned}$$

□

Proof of Theorem 5.2.1. We know now, thanks to Lemma 5.2.2, that the function $z(x,y) = y^{1-\sigma}\frac{\partial w}{\partial y}(x,y)$ is a solution of (5.2.3). But it is well known that problem (5.2.3) has the following explicit solution given in convolution form:

$$z(x,y) = -\int_{\mathbb{R}^N} P(x-\xi,y)f(\xi)d\xi, \quad \text{where} \quad P(x,y) = d_{N,\sigma}\frac{y^{2-\sigma}}{(|x|^2+y^2)^{\frac{N+2-\sigma}{2}}},$$

that is,

$$z(x,y) = -d_{N,\sigma}\int_{\mathbb{R}^N} \frac{y^{2-\sigma}f(\xi)}{(|x-\xi|^2+y^2)^{\frac{N+2-\sigma}{2}}}d\xi.$$

On the other hand, since $z(x,y) = y^{1-\sigma}\frac{\partial w}{\partial y}(x,y)$, we know from the Fundamental Theorem of Calculus that for some function $g = g(x)$ we have

$$w(x,y) = \int_0^y \hat{y}^{\sigma-1}z(x,\hat{y})d\hat{y} + g(x).$$

We assume that $w(x,y) \longrightarrow 0$ as $y \to \infty$ to see that

$$g(x) = -\int_0^\infty \hat{y}^{\sigma-1}z(x,\hat{y})d\hat{y}.$$

Then, we have the following explicit expression for w:

$$\begin{aligned} w(x,y) &= -\int_y^\infty \hat{y}^{\sigma-1}z(x,\hat{y})d\hat{y} \\ &= d_{N,\sigma}\int_y^\infty \hat{y}^{\sigma-1}\int_{\mathbb{R}^N} \frac{\hat{y}^{2-\sigma}f(\xi)}{(|x-\xi|^2+\hat{y}^2)^{\frac{N+2-\sigma}{2}}}d\xi d\hat{y} \\ &= d_{N,\sigma}\int_{\mathbb{R}^N} f(\xi)\int_y^\infty \frac{\hat{y}}{(|x-\xi|^2+\hat{y}^2)^{\frac{N+2-\sigma}{2}}}d\hat{y}d\xi. \end{aligned} \tag{5.2.4}$$

Now we change variables to $v^2 = |x-\xi|^2 + \hat{y}^2$. We get that $\hat{y}d\hat{y} = vdv$ and also that, for $y = 0$, $v = |x-\xi|$. Then, for any $\sigma \in (0,2)$ ($\sigma \in (0,1)$ if $N = 1$) we have obtain

$$\begin{aligned} \lim_{y\to 0} w(x,y) &= d_{N,\sigma}\int_{\mathbb{R}^N} f(\xi)\int_0^\infty \frac{\hat{y}}{(|x-\xi|^2+\hat{y}^2)^{\frac{N+2-\sigma}{2}}}d\hat{y}d\xi \\ &= d_{N,\sigma}\int_{\mathbb{R}^N} f(\xi)\int_{|x-\xi|}^\infty \frac{v}{v^{N+2-\sigma}}dvd\xi \\ &= d_{N,\sigma}\int_{\mathbb{R}^N} f(\xi)\int_{|x-\xi|}^\infty \frac{1}{v^{N+1-\sigma}}dvd\xi \\ &= d_{N,\sigma}\int_{\mathbb{R}^N} f(\xi)\left[\frac{v^{\sigma-N}}{\sigma-N}\right]_{|x-\xi|}^\infty d\xi \\ &= \frac{d_{N,\sigma}}{N-\sigma}\int_{\mathbb{R}^N} \frac{f(\xi)}{|x-\xi|^{N-\sigma}}d\xi, \end{aligned}$$

that is,

$$\lim_{y\to 0} w(x,y) = c_{N,\sigma}\int_{\mathbb{R}^N}\frac{f(x)}{|x-\xi|^{N-\sigma}} = \tilde{d}_{N,\sigma}(-\Delta)^{-\frac{\sigma}{2}}f(x).$$

□

5.2.1 Application: Semi-discrete numerical approximation of the equation $u_t = \nabla\cdot(u^{m-1}\nabla(-\Delta)^{-\frac{\sigma}{2}}u)$

At the end of Chapter 3, a number of interesting open problems are proposed. In the last one we ask for numerical methods for nonlocal parabolic equations in divergence form. One example of this kind of equations is the so-called Porous Medium Equation with Fractional Pressure,

$$\begin{cases} u_t(x,t) = \nabla\cdot(u^{m-1}\nabla(-\Delta)^{-\frac{\sigma}{2}}u), & \text{for } x\in\mathbb{R}^N,\ t\in(0,T),\\ u(x,0) = f(x) & \text{for } x\in\mathbb{R}^N, \end{cases} \tag{5.2.5}$$

for $m\geq 1$ and $\sigma\in(0,2)$. We present here a semi-discrete approach that could be a good starting point for a convergent numerical scheme. Consider a time step that we will denote by Δt and a number of steps J such that $J\Delta t = T$. For $j = 0,\dots,J$ we consider the discretization of the interval $(0,T)$ given by $t_j = j\Delta t$. We will denote by $U_j(x)$ the numerical approximation of $u_j(x) := u(x,t_j)$, that is,

$$u_j(x) := u(x,t_j) \sim U_j(x).$$

In this way, we have a semi-discrete version of the first equation in (5.2.5) given by

$$\frac{U_{j+1}(x)-U_j(x)}{\Delta t} = \nabla\cdot\left((U_j(x))^{m-1}\,\nabla(-\Delta)^{-\frac{\sigma}{2}}U_j(x)\right),\quad \text{for } x\in\mathbb{R}^N, \tag{5.2.6}$$

for $j = 0,\dots,J-1$ where $U_0(x) = f(x)$. At this point, the problem is still non-local, but we can use the technique introduced in Section 5.2 to formulate an equivalent local problem. We proceed as follows: Let W_j be the solution of the following problem:

$$\begin{cases} \nabla\cdot\left(y^{1-\sigma}\nabla W_j\right) = 0 & x\in\mathbb{R}^N,\ y>0,\\ \dfrac{\partial W_j}{\partial y^\sigma}(x,0) = -U_j(x) & x\in\mathbb{R}^N. \end{cases}$$

Then we know by Theorem 5.2.1 that $\lim_{y\to 0^+} W_j(x,y) = \tilde{d}_{N,\sigma}(-\Delta)^{-\frac{\sigma}{2}}U_j(x)$. In this way, we propose the following system of local partial differential equations which is equivalent to (5.2.6):

$$\begin{cases} \nabla \cdot \left(y^{1-\sigma} \nabla W_j\right) = 0 & x \in \mathbb{R}^N,\ y > 0, \\ \dfrac{\partial W_j}{\partial y^\sigma}(x,0) = -U_j(x) & x \in \mathbb{R}^N,\ y = 0, \\ U_{j+1}(x) = U_j(x) + \dfrac{\Delta t}{\tilde{d}_{N,\sigma}} \nabla \cdot \left((U_j(x))^{m-1} \nabla W_j(x,0) \right), & x \in \mathbb{R}^N,\ y = 0, \end{cases} \tag{5.2.7}$$

for $j = 0, \ldots, J-1$ where $U_0(x) = f(x)$. We can discretize in space problem (5.2.8) in order to have a local fully discrete approximation of (5.2.5).

As we announced at the beginning of this subsection, our aim was to propose a numerical approximation of (5.2.5) but we have gone farther on some sense. In particular we have shown that the nonlocal problem (5.2.5) is equivalent to the following system of local partial differential equations:

$$\begin{cases} \nabla \cdot \left(y^{1-\sigma} \nabla w\right) = 0 & x \in \mathbb{R}^N,\ y > 0,\ t > 0, \\ \dfrac{\partial w}{\partial y^\sigma}(x,0,t) = -u(x,t) & x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{\tilde{d}_{N,\sigma}} \nabla \cdot \left((u(x,t))^{m-1} \nabla w(x,0,t) \right), & x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}^N,\ y = 0,\ t = 0. \end{cases} \tag{5.2.8}$$

Note 5.2.1. The technique of formulating a local equivalent problem could be trivially adapted to other nonlocal diffusion models involving inverse fractional laplacians. (M3) and (MG) presented in Chapter 3 are more examples of this kind of equations.

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